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THE FIRST PASSAGE TIME PROBLEM FOR

SIMPLE PHYSICAL SYSTEMS

A THESIS

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SIMPLE PHYSICAL SYSTEMS

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This thesis is dedicated to Sherry.

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SUMMARY

An analysis of the first passage time problem is presented. Lumped parameter models for structural systems are considered, and the input excitation is limited to white noise.

The damped harmonic oscillator is used as a pilot model. Its response to white noise is a two-dimensional Markov process on a position velocity space. This property is basic to the formulation of the first passage time problem that is presented here. Higher order lumped parameter systems also exhibit a Markov response to a white noise excitation, but for these systems the response is Markov on a position-velocity of more than two dimensions. This multi-dimensional Markov property is used in the extension of the present formulation to higher order systems.

Two types of problems are considered for the harmonic oscillator. One is called the fixed start problem. In this problem the oscillator is started from a given position and velocity within a region of safe operation. The second is called the stationary start problem. In this problem the oscillator starts from a randomly located initial point that is distributed in the region of safe operation according to the probability law for the position and velocity of an oscillator which was set into motion at minus infinity and excited by white noise.

Two regions of safe operation are considered. The first has a single position boundary as $x = b$. The second has symmetric boundaries at $x = \pm b$.

In all cases the formulation starts with the derivation of an integral equation that catalogues the sample paths according to their time of first passage. Two unknown functions appear in this equation. The first passage time density function and the conditional density function for the random variable for the velocity of a first passage, given that a first passage occurs. If this conditional density function were known, the integral equation would reduce to a Volterra integral equation of the second kind with the first passage time density as the unknown function.

Several approximations for this conditional density function are discussed. The best of these appears to be an approximation that equates the probability for the velocity of a first passage to the probability of the velocity of an arbitrary crossing of the boundary from within the region. Of course both probabilities are conditioned by the event that a crossing has occurred at the time of interest and by the initial state in the position-velocity space.

Calculations for the first passage time are carried out for both the fixed start and the stationary start problems. Several values of the damping factor of the oscillator and the boundary level are considered. The results for the fixed start problem were compared with estimates obtained by K. L. Chandiramani using a method which he calls diffusion of probability. In this method a numerical simulation of the diffusion of the probability mass in the region of safe operation is carried out and an account is kept (in time) of that mass which escapes through the boundary. Results for the stationary start problem are compared with estimates obtained by R. G. Cook in a Monte Carlo analysis.

Also, calculations were carried out for the mean first passage time for the special case where spring and damping constants are zero. The resulting estimates were compared with a known exact solution. Good agreement was obtained between the method proposed here and the noted results.

In order to illustrate the extension of the present formulation of the first passage time problem to a higher order system a two mass coupled oscillator is considered. A region of safe operation is set by a position boundary for one of the masses, and an integral equation is derived. It is noted that this integral equation involves little more machine calculation than the equation for the oscillator.

CHAPTER I

INTRODUCTION

A first passage time $\tau(\bar{x})$ is the time required for a sample path of a regular, separable stochastic process $\{\bar{X}\}$[1,2] to leave a domain of safe operation Γ starting from a position $\bar{x}$ in Γ.[3] Such a random time can be characterized as a random variable on some probability space by tracing each sample path $\bar{X}_\omega(t)$, which starts at time $t = 0$ at $\bar{X}_\omega(0) = \bar{x}$ in Γ, and observing the time at which it first leaves the region Γ. Mathematically, the following definition may be made [1, p. 625]:[4]

$$\tau_\omega(\bar{x}) = \begin{cases} \sup\left\{ t \;\middle|\; \bar{X}_\omega(t^1) \in \Gamma \text{ for } 0 \le t^1 < t \text{ and } \bar{X}(0) = \bar{x}\right\}, \\ \qquad\qquad \text{for } \omega \notin \Lambda_\Gamma \\ 0, \qquad\qquad \text{for } \omega \in \Lambda_\Gamma \end{cases} \tag{1}$$

where Γ is a Borel set and Λ_Γ is a set of sample functions

1. Appendix A gives a definition of a stochastic process and develops the basic concepts of stochastic analysis (including regularity and separability) that are needed throughout the present work.

2. Bar indicates that the quantity is a vector.

3. In this work Γ is a region in a Euclidean space.

4. Numbers in [ ] refer to references listed in Literature Cited.

which has probability zero.

A probability $Q_{\Gamma}(\bar{x},t)$ can be defined for this random variable as the probability of paths starting from $\bar{x}$ and remaining in Γ at least until time t. It is

$$Q_{\Gamma}(\bar{x},t) = \text{Prob}\left[\tau(\bar{x}) > t\right] \qquad (2)^{5}$$

$1 - Q_{\Gamma}(\bar{x},t)$ is the probability $\text{Prob}\left[\tau(\bar{x}) \le t\right]$. Thus, $1 - Q_{\Gamma}(\bar{x},t)$ is the probability distribution for the random variable $\tau(\bar{x})$, and if $Q_{\Gamma}(\bar{x},t)$ is differentiable the first passage time density function $f_{\Gamma}(\bar{x},t)$ exists. It is

$$f_{\Gamma}(\bar{x},t) = -\frac{\partial Q_{\Gamma}(\bar{x},t)}{\partial t} \qquad (3)$$

Such probabilistic information is extremely important in design considerations in many fields of engineering. For example, in structural fatigue, exceeding a given stress might change the fatigue characteristics of a material. In control theory exceeding a given velocity might place an unacceptable demand on a control system. In communication theory exceeding a voltage might cause an amplifier to saturate thus losing part of a message. In this work idealized models of systems such as physical structures or electronic networks are considered. Such quantities as the stress at a given point or the velocity of an element

5. Prob [ ] is used to denote the probability of occurrence of the event within the brackets.

are representable as the response or components of the response of such a system to a noise excitation. The noise excitation is employed to characterize physical phenomena like combustion in a jet or rocket engine or thermal emission in an electronic network.

The first passage time problem is the problem of expressing functions such as $Q_{\Gamma}(\bar{x},t)$ or $f_{\Gamma}(\bar{x},t)$ in terms of available probabilistic information such as the hierarchy of distributions which specify the process $\{\bar{X}\}$. In actual practice probabilistic information of this type is rather limited. For many systems it is only possible to obtain moments of a few orders for the response process [2, p. 405]. From the classical exclusion series of S. O. Rice [3, p. 70], it should be expected that any formulation of the first passage problem would require rather complete statistical information. Therefore, attention is limited in the present work to a class of processes for which complete statistical information is available. This class may be specified by two statements. First, only processes which can be described by a Langevin stochastic differential equation of finite order or a finite system of such equations are considered [4, p. 271]. As an example, the damped harmonic oscillator may be considered. The Langevin equation is

$$\left\{ \frac{d^2}{dt^2} + \beta \frac{d}{dt} + \omega_o^2 \right\} X_{\tilde{\omega}}(t) = F_{\omega}(t) \tag{4}$$

In this equation $\{F\}$ is the input process, and $\{X\}$ is the response process. The quantities β and ω_o are the damping factor and the natural frequency for the system. Second, only the purely random process, white noise, is allowed as an input excitation.[6]

The first statement means that only lumped parameter models for structures may be considered. Then, the response of a finite lumped parameter system to white noise is a Markov process on a space of one or more dimensions where the components of the multi-dimensional process are the positions and velocities of the elements [4, p. 271].[7] The Markov property makes it possible to give a complete statistical specification of the process from the conditional probability density functions $p(\bar{y},s/\bar{x},r)$ and the probability density function $p(\bar{x},r)$ for $\infty < r < s < \infty$. These probability densities are defined by the equations

$$p(\bar{y},s/\bar{x},r)\ d\bar{y} = \text{Prob}\left[y < \bar{X}(s) \leq \bar{y} + d\bar{y} \,/\, \bar{X}(r) = \bar{x}\right]$$

$$p(\bar{x},r)\ d\bar{x} = \text{Prob}\left[\bar{x} < \bar{X}(r) \leq \bar{x} + d\bar{x}\right] \tag{5}$$

where $\bar{y} < \bar{X}(s) \leq \bar{y} + d\bar{y}$ means that every component $X^i(s)$ satisfies the relationship $y^i < X^i(s) \leq y^i + dy^i$.

A famous system which satisfies these two restrictions is the viscously damped Brownian particle. The Langevin equation for this

6. The use of the purely random process and the Langevin equation in the present work is discussed in detail in Appendix B.

7. It would be possible to consider a system where second, third, and higher order derivatives are components of the response process; however, this is not done in the present work.

system is

$$\left\{\frac{d}{dt} + \beta\right\} \quad X_{\tilde{\omega}}(t) = F_{\omega}(t)$$

In this case $\{X\}$ is a one-dimensional Markov process describing the velocity of the particle. It is called the Ornstein-Uhlenbeck process [5] The properties of this process including results on first passage time probabilities are available in the classical literature on stochastic processes [6] and [7].

One-dimensional Markov processes with continuous sample paths such as the Ornstein-Uhlenbeck process are well understood. In fact, Siegert gives a formula for the first passage probability density $f_{\Gamma}(\overline{x},t)$, [6, p. 618]. Multi-dimensional Markov processes with continuous sample paths exist which can be simplified if their projection processes are themselves individually Markov, and first passage and other probabilistic information can be obtained [8, pp. 205-209] and [9, p. 82]. In the case of the harmonic oscillator such a simplification is not possible. The response process is a two-dimensional Markov process on a position-velocity space, $\{\overline{X}\} = \{(X,\dot{X})\}$. The process $\{X\}$ describes the position, and the process $\{\dot{X}\}$ describes the velocity. These projections are not individually Markov when considered by themselves. It has been noted by researchers such as Kac, Sigert, and others that this fact substantially complicates the first passage problem [10, p. 35]. Difficulties associated with other lumped parameter systems which can be characterized by still higher dimensional Markov processes are similar to those associated with the harmonic oscillator. For this reason the harmonic oscillator is used as a pilot model, with the hope of extending results obtained for it to other systems.

The response of the damped harmonic oscillator to a white noise excitation is analyzed in detail in Appendix B. From the analysis given there the following facts are apparent:

(1) The response process for the harmonic oscillator is a two-dimensional Markov process in a position-velocity space.

(2) It is a stationary Gaussian process.

(3) Explicit forms for $p(y,\dot{y},s/x,\dot{x},r)$ and $p(x,\dot{x},r)$ exist. The first passage problem in this case is the problem of using this information to develop a procedure for the evaluation of the probability $Q_{\Gamma}(\overline{x},t)$ or the probability density $f_{\Gamma}(\overline{x},t)$ for a given region Γ in a two-dimensional euclidean space. Note that $\overline{x}$ is the vector $(x,\dot{x})$.

The region Γ could be as intricate as one wishes. However, if a Brownian particle on a one-dimensional euclidean space [5] is considered and the analyses of the problem of a particle remaining in an interval $(-\infty, b)$ where b is finite and the problem of a particle remaining in the interval (a,b) where a and b are finite are examined [7, pp. 328 and 329], it is seen that the analysis of a one-dimensional first passage problem is dependent on the type of region being considered.

8. Since the process is stationary $p(y,\dot{y},s/x,\dot{x},r)$ is a function of $s-r$ only. Many places in this work this function is written $p(y,\dot{y},t/x,\dot{x})$ where $t = s - r$. Also, $p(x,\dot{x},r)$ is independent of time and is written $p(x,\dot{x})$.

The unbounded lower limit on x greatly simplifies the procedure. In the case of the two-dimensional problem being considered in the present research, possibly the simplest region to consider would be the semi-infinite plane below a line $x = b$, i.e.,

$$\Gamma_1 = \left\{(x,\dot{x})/ -\infty < x \leq b \text{ and } -\infty < \dot{x} < \infty\right\}$$

The main objective of the present study is a formulation of the first passage problem for this region. Also, another region of practical interest is considered. It is

$$\Gamma_2 = \left\{(x,\dot{x}) \quad a < x \leq b \text{ and } -\infty < \dot{x} < \infty\right\}$$

Here a and b are finite numbers.

The fact that the backward Fokker-Planck equation may be solved to obtain the conditional density $p(y,\dot{y},s/x,\dot{x},r)$ is discussed in Appendix B. M. Kac illustrates [11, p. 38] that $Q_{\Gamma}(x,\dot{x},t)$ must satisfy the backward Fokker-Planck equation on the region Γ in a position-velocity space. To obtain this relationship it is advantageous to define a new conditional density function $p_{\Gamma}(y,\dot{y},s/x,\dot{x},r)$ by the equation

$$p_{\Gamma}(y,\dot{y},s/x,\dot{x},r)\,d\dot{y}dy = \text{Prob}\left[\begin{array}{c} (X(\tau),\dot{X}(\tau)) \epsilon \Gamma \;,\quad \tau \epsilon (s,r) \\ y < X(s) \le y + dy \\ \\ \dot{y} < \dot{X}(s) \le \dot{y} + d\dot{y} \end{array} \middle| \begin{array}{c} X(r) = x \\ \\ \dot{X}(r) = \dot{x} \end{array}\right] \tag{6}$$

The density function describes the transition of sample paths which remain in the region from time r to time s. It is the equivalent for the region Γ of the function $p(y,\dot{y},s/x,\dot{x},r)$ which describes the transition of sample paths in the whole position-velocity space. The importance of $p_{\Gamma}(y,\dot{y},s/x,\dot{x},r)$ is obvious. By setting $r=0$ it is possible to obtain the basic function $Q_{\Gamma}(x,\dot{x},t)$ by integrating $p_{\Gamma}(y,\dot{y},s/x,\dot{x},r)$ over Γ on the variables y and $\dot{y}$. A higher order density function may be defined in a manner analogous to equation (6). Thus

$$p_{\Gamma}(y,\dot{y},\eta;z,\dot{z},s/x,\dot{x},r)\; dzd\dot{z}dyd\dot{y} = \text{Prob}\left[\begin{array}{c} (X(\tau),\dot{X}(\tau)) \epsilon \Gamma \;,\quad \tau \epsilon (r,s) \\ y < X(\eta) \le y + dy \\ \dot{y} < \dot{X}(\eta) \le \dot{y} + d\dot{y} \\ z < X(s) \le z + dy \\ \dot{z} < \dot{X}(s) \le \dot{z} + d\dot{z} \end{array} \middle| \begin{array}{c} X(r) = x \\ \dot{X}(r) = \dot{x} \end{array}\right]$$

The points $(x,\dot{x})$, $(y,\dot{y})$, and $(z,\dot{z})$ are understood to be in Γ, and the times satisfy the relationship $-\infty < r < \eta < s < \infty$. Because the process is Markov, the relationship

$$p_{\Gamma}(y,\dot{y},\eta;z,\dot{z},s/x,\dot{x},r)\, dz d\dot{z} dy d\dot{y} =$$
$$p_{\Gamma}(z,\dot{z},s/y,\dot{y},\eta) p_{\Gamma}(y,\dot{y},\eta/x,\dot{x},r)\, dz d\dot{z} dy d\dot{y} \tag{7}$$

holds.

Equation (7) may be integrated on y and $\dot{y}$ over the region Γ. Since $(X(\tau), \dot{X}(\tau))$ is required to remain in Γ for all τ between r and s, the event that $(X(\eta), \dot{X}(\eta)) \in \Gamma$ is a certain event. Thus

$$p_{\Gamma}(z,\dot{z},s/x,\dot{x},r) = \iint_{\Gamma} p_{\Gamma}(z,\dot{z},s/y,\dot{y},\eta)\, p_{\Gamma}(y,\dot{y},\eta/x,\dot{x},r)\, dy d\dot{y} \tag{8}$$

This is a Chapman-Kolmogorov equation for the region Γ. If Γ is the whole position-velocity space, this equation becomes the standard Chapman-Kolmogorov equation.

If $p_{\Gamma}(z,\dot{z},s/y,\dot{y},\eta)$ is restricted to the class of functions which have Taylor series expansions about $(y,\dot{y}) = (x,\dot{x})$, equation (8) yields

$$p_{\Gamma}(z,\dot{z},s/x,\dot{x},r) - p_{\Gamma}(z,\dot{z},s/x,\dot{x},\eta) \iint_{-\infty}^{\infty} p_{\Gamma}(y,\dot{y},\eta/x,\dot{x},r)\, dy d\dot{y} =$$

$$\left\{ \begin{array}{l} \left[\frac{\partial}{\partial x} p_{\Gamma}(z,\dot{z},s/x,\dot{x},\eta) \iint_{-\infty}^{\infty} (y - x)\, p_{\Gamma}(y,\dot{y},\eta/x,\dot{x},r)\, dy d\dot{y}\right] \\ + \left[\frac{\partial}{\partial \dot{x}} p_{\Gamma}(z,\dot{z},s/x,\dot{x},\eta) \iint_{-\infty}^{\infty} (\dot{y} - \dot{x})\, p_{\Gamma}(y,\dot{y},\eta/x,\dot{x},r)\, dy d\dot{y}\right] \\ + \left[\frac{\partial^2}{2\partial x^2} p_{\Gamma}(z,\dot{z},s/x,\dot{x},\eta) \iint_{-\infty}^{\infty} (y - x)^2\, p_{\Gamma}(y,\dot{y},\eta/x,\dot{x},r)\, dy d\dot{y}\right] \\ + \left[\frac{\partial^2}{2\partial \dot{x}^2} p_{\Gamma}(z,\dot{z},s/x,\dot{x},\eta) \iint_{-\infty}^{\infty} (\dot{y} - \dot{x})^2\, p_{\Gamma}(y,\dot{y},\eta/x,\dot{x},r)\, dy d\dot{y}\right] \\ + \left[\frac{\partial^2}{\partial x \partial \dot{x}} p_{\Gamma}(z,\dot{z},s/x,\dot{x},\eta) \iint_{-\infty}^{\infty} (\dot{y} - \dot{x})(y - x)\, p_{\Gamma}(y,\dot{y},\eta/x,\dot{x},r)\, dy d\dot{y}\right] \\ + \ldots \end{array} \right. \tag{9}$$

The limits of integration in this equation have been extended from Γ to the whole position-velocity space by defining $p_{\Gamma}(y,\dot{y},\eta/x,\dot{x},s)$ to be zero for $(y,\dot{y}) \notin \Gamma$.

In order to show that equation (9) yields the Fokker-Planck backward equation both sides of (9) must be divided by $\eta - r$ and the limit taken as $\eta \rightarrow r^{+}$. The limits in equation (9) can be evaluated using the fact that the sample functions are continuous with probability one (See Appendix B). This fact implies that

$$\lim_{\eta \to r^+} \iint_{-\infty}^{\infty} p_\Gamma (y,\dot{y},\eta/x,\dot{x},r)\, dyd\dot{y} = \lim_{\eta \to r^+} \iint_{-\infty}^{\infty} p\, (y,\dot{y},\eta/x,\dot{x},r)\, dyd\dot{y} = 1$$

and for $m, n = 0,1,\ldots$ and $m + n \geq 1$

$$\lim_{\eta \to r^+} \frac{1}{\eta - r} \iint_{-\infty}^{\infty} (y - x)^n\, (\dot{y} - \dot{x})^m\, p_\Gamma\, (y,\dot{y},\eta/x,\dot{x},r)\, dyd\dot{y} =$$

$$\lim_{\eta \to r^+} \frac{1}{\eta - r} \iint_{-\infty}^{\infty} (y - x)^n\, (\dot{y} - \dot{x})^m\, p(y,\dot{y},\eta/x,\dot{x},r)\, dyd\dot{y} = A_{m,n}\, (x,\dot{x},r)$$

The parameter $A_{m,n}\,(x,\dot{x},r)$ is a special form of conditional moment. Calculations for $A_{m,n}\,(x,\dot{x},r)$ are given in Appendix B. For $m + n \geq 3$, $A_{m,n}\,(x,\dot{x},r)$ equals zero, and equation (9) yields

$$\left\{ \frac{\partial}{\partial r} + L_2^* \right\} \; p_\Gamma\, (y,\dot{y},s/x,\dot{x},r) = 0 \tag{10}$$

where L_2^* is

$$L_2^* = A_{1,0}(x,\dot{x},r) \frac{\partial}{\partial \dot{x}} + A_{0,1}\,(x,\dot{x},r) \frac{\partial}{\partial x} + \frac{1}{2} A_{2,0}(x,\dot{x},r) \frac{\partial^2}{\partial x^2}$$

$$+ \frac{1}{2} A_{0,2}(x,\dot{x},r) \frac{\partial^2}{\partial \dot{x}^2} + A_{1,1}(x,\dot{x},r) \frac{\partial^2}{\partial x \partial \dot{x}}$$

The above discussion illustrates the fact that $p_{\Gamma}(y,\dot{y},s/x,\dot{x},r)$ satisfies the backward Fokker-Planck equation. The application of conventional methods to the problem of seeking $p_{\Gamma}(y,\dot{y},s/x,\dot{x},r)$ as a solution to the backward equation would require the specification of initial and boundary conditions [12, pp. 617-622]. The initial condition follows from the sample path continuity property. It is

$$\lim_{\eta \to r^+} p_{\Gamma}(y,\dot{y},s/x,\dot{x},r) = \delta(y - x) \quad \delta(\dot{y} - \dot{x}) \tag{11}$$

The boundary conditions follow from consideration of the motion of the sample paths in a position-velocity space. Note that sample paths moving in this space are oriented as shown in Figure 1. Consider the region Γ_1.

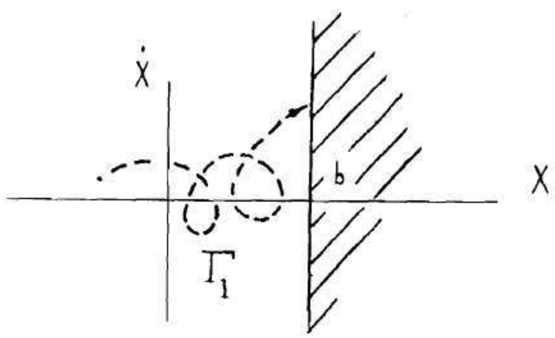

Figure 1. The Region Γ_1 and a Sample Function.

As the initial position x tends to minus infinity or the initial velocity $\dot{x}$ tends to plus or minus infinity, the probability $p_{\Gamma_1}(y,\dot{y},s/x,\dot{x},r)$ should vanish for the same reason that $p(y,\dot{y},s/x,\dot{x},r)$ vanishes (See Appendix B). If a sample path which starts from a position arbitrarily close to the level b is considered, what happens to it depends on its starting velocity. If the path starts with a positive velocity, it will almost certainly be absorbed instantaneously. The probability statement corresponding to this is

$$\lim_{x \to b^-} p_{\Gamma_1}(y,\dot{y},s/x,\dot{x},r) = 0, \quad s > r \tag{12}$$

If the path starts with a negative velocity, it cannot even approach the boundary until its velocity becomes positive. Therefore, nothing can be said about instantaneous absorption. Hence, no apriori probability statement can be made about

$$\lim_{x \to b^-} p_{\Gamma_1}(y,\dot{y},s/x,\dot{x},r), \quad s > r \tag{13}$$

A similar boundary condition can be derived for Γ_2.

The fact that $Q_{\Gamma}(x,\dot{x},t)$ satisfies the backward equation can be deduced from the properties of $p_{\Gamma}(y,\dot{y},s/x,\dot{x},r)$. The moments $A_{m,n}(x,\dot{x},r)$ in the backward equation are not explicit functions of time. Thus, $p_{\Gamma}(y,\dot{y},s/x,\dot{x},r)$ is a function of the time difference s-r only, and it is possible to write

$$p_{\Gamma}(y,\dot{y},s/x,\dot{x},r) = p_{\Gamma}(y,\dot{y},t/x,\dot{x})$$

where $t = s - r$. The backward equation becomes

$$\left\{\frac{\partial}{\partial t} - L_2^*\right\} p_{\Gamma} = 0$$

Equation (8) can be rewritten in the form

$$p_{\Gamma}(z,\dot{z},t/x,\dot{x}) = \iint_{\Gamma} p_{\Gamma}(z,\dot{z},t-\Delta t/y,\dot{y})\, p_{\Gamma}(y,\dot{y},\Delta t/x,\dot{x})\, dy d\dot{y}$$

Integrating over Γ in z and $\dot{z}$ yields

$$Q_{\Gamma}(x,\dot{x},t) = \iint_{\Gamma} Q_{\Gamma}(y,\dot{y},t-\Delta t)\, p_{\Gamma}(y,\dot{y},\Delta t/x,\dot{x})\, dy d\dot{y}$$

If it is assumed that $Q_{\Gamma}(y,\dot{y},t-\Delta t)$ is expandable about $(y,\dot{y}) = (x,\dot{x})$, it may be concluded that $Q_{\Gamma}(x,\dot{x},t)$ satisfies the backward equation. The analysis is similar to the work which shows that $p_{\Gamma}(y,\dot{y},s/x,\dot{x},r)$ satisfies the backward equation.

Initial and boundary conditions can be discussed for $Q_{\Gamma}(x,\dot{x},t)$ with the aid of (11), (12), and (13), and the relationship

$$Q_{\Gamma}(x,\dot{x},t) = \iint_{\Gamma} p_{\Gamma}(y,\dot{y},t/x,\dot{x})\, dy d\dot{y} \tag{14}$$

The initial condition is

$$\lim_{t \to 0^+} Q_{\Gamma}(x,\dot{x},t) = \lim_{t \to 0^+} \iint_{\Gamma} p_{\Gamma}(y,\dot{y},t/x,\dot{x})\, dy d\dot{y} =$$

$$\lim_{t \to 0^+} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(y,\dot{y},t/x,\dot{x})\, dy d\dot{y} = 1$$

or

$$\lim_{t \to 0^+} Q_{\Gamma}(x,\dot{x},t) = 1 \tag{15}$$

To obtain a boundary condition for the region Γ_1, take the limit as x tends to b from the left in equation (14). Thus for fixed $t > 0$, [9]

$$\lim_{x \to b^-} Q_{\Gamma_1}(x,\dot{x},t) = \lim_{x \to b^-} \iint_{\Gamma_1} p_{\Gamma_1}(y,\dot{y},t/x,\dot{x})\, dy d\dot{y} =$$

$$= \iint_{\Gamma_1} \lim_{x \to b^-} p_{\Gamma_1}(y,\dot{y},t/x,\dot{x})\, dy d\dot{y} = \tag{16}$$

$$= \begin{cases} 0, \text{ if } \dot{x} > 0 \\ \text{(not possible to specify apriori if } \dot{x} < 0) \end{cases}$$

Comments

The central theme of the present work is the formulation of the first passage time problem for a mechanical system which is excited by white noise. The primary example is the damped harmonic oscillator. A differential equation formulation is presented in this chapter. Some other formulations of this problem have been presented in the literature,

9. The inequality $0 \leq p_{\Gamma_1}(y,\dot{y},t/x,\dot{x}) \leq p(y,\dot{y},t/x,\dot{x})$ is valid for $t > 0$. This inequality can be used to construct a function of y and $\dot{y}$ which dominates $p_{\Gamma_1}(y,\dot{y},t/x,\dot{x})$ for x in some neighborhood of b. Therefore, the Lebesgue dominated convergence theorem [13, p. 125] may be used to infer the validity of equation (16).

and it seems appropriate to comment on them at this juncture.

S. O. Rice has suggested approximating the first passage time density function by an extremely complex multiple integral [3, p. 70]. K. L. Chandiramani has reduced the difficulty of evaluating this integral somewhat by using the Markov property of the response of the damped harmonic oscillator [14, p. 15]. Even with his simplification this formulation requires a great deal of computer storage and time, and it requires a rather elaborate program [14, pp. 41-54]. Also, an extension to a more complex structural system would enlarge the computer requirements a great deal.

Other methods have been suggested by J. R. Rice [15] and Y. K. Lin [16]. Both authors work depends on assumptions that simplify inclusion, exclusion series similar to the classical procedure suggested by S. O. Rice [3, p. 70]. In both works the justification of the basic assumptions is heuristic in nature, and no rigorous justification of the concepts involved is available.

The differential equation approach seems very appealing in that it makes an association between the backward equation, the Markov property, and the first passage probability functions. The fact that boundary data is not specified on the whole boundary (Consider (16)) had generated some questions concerning existence and uniqueness of the solution for the boundary value problem composed of the backward Fokker-Planck equation and conditions (15) and (16) [17, p. 339] and [11, p. 39]. Yang and Sihnozuka [18, p. 393-394] have indicated that Fichara's theory of eliptic-parabolic equations [19, p. 87-120] can be used to show that such boundary value problems are well posed. Franklin and Rodemich

[20, pp. 683-697] found the mean first passage time for a Γ_2 type region where $a = -b$ in the special case where $\beta = 0$ and $\omega_o = 0$ by solving a boundary value problem that is related to the one that is derived here for $Q_{\Gamma_1}(x,\dot{x},t)$.

Since it appears that the boundary value problem that is presented here has a unique solution, it would be possible to obtain a numerical solution by some type of difference technique. However, in the present research a different line of attack is chosen. An integral equation which the first passage time density function must satisfy is formulated in Chapter II. The remaining work is devoted to the problem of obtaining approximate solutions for this integral equation.

CHAPTER II

AN INTEGRAL EQUATION SATISFIED BY THE FIRST PASSAGE TIME PROBABILITY DENSITY FUNCTION

In Chapter I it was pointed out that the response of the harmonic oscillator to white noise is a two-dimensional Markov process in a position-velocity space. Also, it is known that this response is a stationary Gaussian process; and expressions exist for the density functions $p(x,\dot{x},t)$ and $p(y,\dot{y},s/x,\dot{x},r)$ which completely specify the process. The function $p(y,\dot{y},s/x,\dot{x},r)$ satisfies forward and backward Fokker-Planck equations. In addition, the probability $Q_{\Gamma_1}(x,\dot{x},t)$ of remaining in the region Γ_1 has been shown to satisfy the backward equation.

The present chapter is devoted to the development of an integral equation satisfied by the first passage time probability density function. Attention is restricted to the region Γ_1 with boundary at $x = b$.

Let $\dot{X}(\tau(x,\dot{x}))$ be the velocity with which a sample function makes its first passage from the region Γ_1.[10] Note that this random variable

10. Quite often functions of a random variable do not themselves fulfill the requirements of a random variable [13, p. 578]. However, it is possible to show that $\dot{X}(\tau(x,\dot{x}))$ is a random variable for the processes that are considered in this work. A proof of this is given in Appendix C.

is not the same as $\dot{X}(t)$, the velocity of the process.

Since $\tau(x,\dot{x})$ and $X(\tau(x,\dot{x}))$ are random variables, they possess a joint probability distribution function. It is

$$G_{\Gamma_1}(\eta,\dot{y}) = \text{Prob}\left\{\tau(x,\dot{x}) \le \eta,\ \dot{X}(\tau(x,\dot{x})) \le \dot{y}\right\}$$

Note that

$$1 - Q_{\Gamma_1}(x,\dot{x},\eta) = G_{\Gamma_1}(\eta,\infty)$$

$G_{\ 1}(\eta,\infty)$ is thus the probability distribution function for the first passage time. Now, letting

$$H_{\Gamma_1}(\dot{y},\eta;z,\dot{z},t/x,\dot{x},0)$$

be the conditional distribution function for the four random variables $\dot{X}(\tau(x,\dot{x}))$, $\tau(x,\dot{x})$, $X(t)$ assuming

$$X(0) = x \text{ and } \dot{X}(0) = \dot{x}$$

and letting $H_{\Gamma_1}(z,\dot{z},t/x,\dot{x},0;\dot{y},\eta)$ be the conditional distribution function of $X(t)$ and $\dot{X}(t)$ assuming

$$X(0) = x;\ \dot{X}(0) = \dot{x},\ \dot{X}(\tau(x,\dot{x})) = \dot{y} \text{ and } \tau(x,\dot{x}) = \eta$$

the following integral equation may be written:

$$H_{\Gamma_1}(\dot{y},\eta;z,\dot{z},t/x,\dot{x},0) = \int_0^{\eta}\int_0^{\dot{y}} H_{\Gamma_1}(z,\dot{z},t/x,\dot{x},0;\dot{y}',\eta')g_{\Gamma_1}(\eta',\dot{y}')\, d\dot{y}'d\eta' \qquad (17)$$

where

$$g_{\Gamma_1}(\eta,\dot{y}) = \frac{d^2\, G_{\Gamma_1}(\eta,\dot{y})}{d\eta d\dot{y}}$$

In the case of the harmonic oscillator the specification of state $\dot{X}(\tau(x,\dot{x})) = \dot{y}$ and $\tau(x,\dot{x}) = \eta$ is equivalent to the specification of state $(X(\eta), \dot{X}(\eta)) = (b,\dot{y})$.[11]

Thus

$$H_{\Gamma_1}(z,\dot{z},t/x,\dot{x},0;\dot{y},\eta) = F\,(z,\dot{z},t/x,\dot{x},0;b,\dot{y},\eta) = F\,(z,\dot{z},t/b,\dot{y},\eta)$$

where the last equality follows from the Markov property. Substituting this into equation (17) and setting $\eta = t$ and taking the limit as $\dot{y}$ tends to infinity gives

$$H_{\Gamma_1}(\infty,t;z,\dot{z},t/x,\dot{x},0) = \int_0^{t}\int_0^{\infty} F(z,\dot{z},t/b,\dot{y},\eta)\quad g_{\Gamma_1}(\eta,\dot{y})\, d\dot{y}d\eta \qquad (18)$$

11. D. Ray has given an example of a Markov process for which these specifications are not equivalent [21, sec. IV]. Also, under a set of hypotheses more general than the set considered when seeking $p(y,\dot{y},s/x,\dot{x},r)$ as a solution of the forward and backward Fokker-Planck equations, he proves that these two specifications are equivalent for stationary processes [21, pp. 467-469].

If $z \leq b$, the probability of a first passage at a time less than or equal to t must equal the probability of either having a first passage or not having one minus the probability of not having one. That is

$$H_{\Gamma_1}(\infty, t; z, \dot{z}, t/x, \dot{x}, 0) = F(z, \dot{z}, t/x, \dot{x}, 0) - F_{\Gamma_1}(z, \dot{z}, t/x, \dot{x}, 0)$$

Finally, equation (18) may be written as

$$F(z, \dot{z}, t/x, \dot{x}, 0) = F_{\Gamma_1}(z, \dot{z}, t/x, \dot{x}, 0) \; +$$

$$+ \int_0^t \int_0^\infty F(z, \dot{z}, t/b, \dot{y}, \eta) \, g_{\Gamma_1}(\dot{y}/\eta) \, f_{\Gamma_1}(x, \dot{x}, \eta) \, d\dot{y} d\eta \qquad (19)$$

Differentiating equation (19) with respect to t, taking z equal to b, taking the limit as $\dot{z}$ tends to infinity, and recalling that

$$\frac{\partial}{\partial t} G_{\Gamma_1}(t, \infty) = f_{\Gamma}(x, \dot{x}, t)$$

and

$$G_{\Gamma_1}(t, \infty) = 1 - F_{\Gamma_1}(b, \infty, t/x, \dot{x}, 0)$$

the following integral equation is obtained:[12]

$$\frac{\partial}{\partial t} F(b,\infty,t/x,\dot{x},0) = -f_{\Gamma_1}(x,\dot{x},t)$$

$$+ \int_0^t \int_0^\infty \frac{\partial}{\partial t} F(b,\infty,t/b,\dot{y},\eta) g_{\Gamma_1}(\dot{y}/\eta)\, f_{\Gamma_1}(x,\dot{x},\eta)\, d\dot{y}\, d\eta \tag{20}$$

The function $F(b,\infty,t/x,\dot{x},0)$ is the probability of sample paths being in Γ_1 at time t, given that they start at $(x,\dot{x})$. Thus $\frac{\partial}{\partial t} F(b,\infty,t/x,\dot{x},0)$ is the rate of increase of this probability or net flux of sample paths into the region Γ_1. This function can be split up into two parts. They are the expected number of negative crossings of b per unit time $N^-(t/x,\dot{x},0)$ (flux into Γ_1) and the expected number of positive crossings per unit time $N^+(t/x,\dot{x},0)$ (flux out of

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12. $F(b,\infty,t/b,\dot{y},\eta)$ is not continuous at $\eta = t$. $\lim_{\eta \to t^-} F(b,\infty,t/b,\dot{y},\eta) = 0$ but $F(b,\infty,t/b,\dot{y},t) = 1$. Thus $\frac{\partial}{\partial t} F(b,\infty,t/b,\dot{y},\eta)$ does not exist at $\eta = t$; and conventional theorems for differentiating under the integral are not applicable. The partial derivative does exist and can be dominated by an integrable function of $\dot{y}$ and η $(0<\dot{y}<\infty$ and $0<\eta<t)$ except for $\dot{y}$ and η in a neighborhood of $\dot{y} = 0$, $\eta = t$. This partial derivative has an integrable singularity at $\dot{y} = 0$, $\eta = t$. The dominated convergence theoren can be used to accomplish the differentiation outside the neighborhood of $\dot{y} = 0$, $\eta = t$. The contribution which results from the integration over this neighborhood can be shown to approach zero as the aera of the neighborhood is shrunk to zero. Thus equation (20) results.

Γ_1).[13] Thus

$$\frac{\partial}{\partial t} F(b,\infty,t/x,\dot{x},0) = N^{-}(t/x,\dot{x},0) - N^{+}(t/x,\dot{x},0)$$

$N^{+}(t/x,\dot{x},0)$ and $N^{-}(t/x,\dot{x},0)$ may be expressed in terms of S. O. Rice's formula [3, p. 58]. They are

$$N^{+}(t/x,\dot{x},0) = \int_0^\infty \dot{w}\ p(b,\dot{w},t/x,\dot{x},0)\ d\dot{w}$$

$$N^{-}(t/x,\dot{x},0) = -\int_{-\infty}^0 \dot{w}\ p(b,\dot{w},t/x,\dot{x},0)\ d\dot{w}$$

Also, $\frac{\partial}{\partial t} F(b,\infty,t/b,\dot{y},\eta)$ may be split up in the same way. Thus equation (20) may be written as

$$\left.\begin{matrix} N^{+}(t/x,\dot{x},0) + \\ \\ - N^{-}(t/x,\dot{x},0) \end{matrix}\right\} = \left\{\begin{matrix} f_{\Gamma_1}(x,\dot{x},t) + \\ + \int_0^t \int_0^\infty N^{+}(t/b,\dot{y},\eta)\ g_{\Gamma_1}(\dot{y}/\eta)\ f_{\Gamma_1}(x,\dot{x},\eta)d\dot{y}d\eta + \\ - \int_0^t \int_0^\infty N^{-}(t/b,\dot{y},\eta)\ g_{\Gamma_1}(\dot{y}/\eta)\ f_{\Gamma_1}(x,\dot{x},\eta)d\dot{y}d\eta \end{matrix}\right. \tag{21}$$

13. A positive crossing is a crossing of the level b with a positive slope; a negative crossing is a crossing of b with a negative slope.

If $z > b$, $F_{\Gamma}(z,\dot{z},r/x,\dot{x},0)$ equals $F_{\Gamma_1}(b,\dot{z},t/x,\dot{x},0)$ because it is impossible for a sample path to obtain a position greater than b without having a first passage. Thus

$$H_{\Gamma_1}(\infty;t;z,\dot{z},t/x,\dot{x},0) - H_{\Gamma_1}(\infty,t;b,\dot{z},t/x,\dot{x},0) =$$

$$= \begin{cases} F(z,\dot{z},t/x,\dot{x},0) - F_{\Gamma_1}(z,\dot{z},t/x,\dot{x},0) + \\ - F(b,\dot{z},t/x,\dot{x},0) + F_{\Gamma_1}(b,\dot{z},t/x,\dot{x},0) \end{cases}$$

$$= \quad F(z,\dot{z},t/x,\dot{x},0) - F(b,\dot{z},t/x,\dot{x},0)$$

This difference may be expressed using equation (18). The result is

$$F(z,\dot{z},t/x,\dot{x},0) - F(b,\dot{z},t/x,\dot{x},0) =$$

$$= \int_0^t \int_0^\infty F(z,\dot{z},t/b,\dot{y},\eta) - F(b,\dot{z},t/b,\dot{y},\eta) \quad g_{\Gamma_1}(\eta,\dot{y}) \; d\dot{y}d\eta$$

Differentiating both sides with respect to z and $\dot{z}$ yields

$$p(z,\dot{z},t/x,\dot{x},0) = \int_0^t \int_0^\infty p(z,\dot{z},t/b,\dot{y},\eta) \; g_{\Gamma_1}(\eta,\dot{y}) \; d\dot{y}d\eta$$

Taking the limit as $z \to b$ from above for $\dot{z} < 0$, multiplying by $\dot{z}$ and integrating from minus infinity to zero yields

$$\int_{-\infty}^{0} -\dot{z}\ p(b,\dot{z},t/x,\dot{x},0)\ d\dot{z} = \int_{0}^{t}\int_{0}^{\infty}\int_{-\infty}^{0} -\dot{z}\ p(b,\dot{z},t/b,\dot{y},\eta)\ g_{\Gamma_1}(\eta,\dot{y})\ d\dot{z}d\dot{y}d\eta$$

Writing this in terms of $N^-(t/x,\dot{x},0)$ and $N^+(t/b,\dot{y},\eta)$ results in

$$N^-(t/x,\dot{x},0) = \int_{0}^{t}\int_{0}^{\infty} N^-(t/b,\dot{y},\eta)\ g_{\Gamma_1}(\dot{y}/\eta)\ f_{\Gamma_1}(x,\dot{x},\eta)\ d\dot{y}d\eta \qquad (22)$$

Subtracting equation (22) from (21) yields

$$N^+(t/x,\dot{x},0) = f_{\Gamma_1}(x,\dot{x},t) + \int_{0}^{t}\int_{0}^{\infty} N^+(t/b,\dot{y},\eta)\ g_{\Gamma_1}(\dot{y}/\eta)\ f_{\Gamma_1}(x,\dot{x},\eta)\ d\dot{y}d\eta \qquad (23)$$

$N^+(t/x,\dot{x},0)$ is a known function; if $g_{\Gamma_1}(\dot{y}/\eta)$ were known, equation (23) would contain only the unknown function $f_{\Gamma_1}(x,\dot{x},\eta)$. Approximations for $g_{\Gamma_1}(\dot{y}/\eta)$ will be given in Chapter III. Then, the numerical solution of equation (23) for $f_{\Gamma_1}(x,\dot{x},t)$ will be discussed in Chapter IV.

CHAPTER III

METHODS OF APPROXIMATION AND THEIR PROBABILISTIC IMPLICATIONS

The approach that is discussed in the present chapter is indicated at the end of Chapter II. Approximations are developed for the conditional density function $g_{\Gamma_1}(\dot{y}/\eta)$. With these approximations equation (23) can be solved by a simple numerical procedure to obtain estimates of the first passage time density function $f_{\Gamma_1}(x,\dot{x},t)$.

Three methods of approximation are proposed here. The first provides a link between this work and the independent waiting time models which have application in many problems in probability [7, Sec. VI. 6-7]. The second method provides a connection between this work and the study of the first passage problem conducted by J. R. Rice [15]. The third approximation results from an attempt to more meaningfully employ the probabilistic properties of the process considered in the present work.

Some conditions which $g_{\Gamma_1}(\dot{y}/\eta)$ must satisfy are given in this chapter. Although they will not determine $g_{\Gamma_1}(\dot{y}/\eta)$ exactly, they will show that of the approximations considered here the third is most appropriate for the harmonic oscillator.

A method for determining $g_{\Gamma_1}(\dot{y}/\eta)$ does not appear to be at hand. Therefore, some check on the approximations to $g_{\Gamma_1}(\dot{y}/\eta)$ presented here is needed. This is done in Chapter IV where estimates for the first passage time density function that result from the approximations given here are compared with the results of other approximate methods such as Monte Carlo techniques and the multiple integration method suggested by K. L. Chandiramani [14]. Also, a comparison between the exact solution, obtained by Franklin and Rodemich [20, pp. 683-693], for the mean first passage time for the special case where β and ω_0 are zero and the estimates which result from the approximations that are presented here is given in Chapter IV.

Independent Crossing Model

If the damping in the oscillator is large, the density function $p(b,\dot{w},t/b,\dot{y},\eta)$ will approach $p(b,\dot{w})$ rapidly as $t - \eta$ becomes large. Thus

$$\lim_{t-\eta \to \infty} N^{+}(t/b,\dot{y},\eta) = \int_0^{\infty} \lim_{t-\eta \to \infty} \dot{w}\, p(b,\dot{w},t/b,\dot{y},\eta)\, d\dot{w}$$

$$= \int_0^{\infty} \dot{w}\, p(b,\dot{w})\, d\dot{w}$$

This limiting value will be denoted by N^{+} in the present research. That is

$$N^{+} = \int_0^{\infty} \dot{w}\, p(b,\dot{w})\, d\dot{w}$$

In the above circumstance it may be appropriate to use N^{+} to approximate $N^+(t/b,\dot{y},\eta)$. N^+ is not a function of t or η; so that, this approximation may be viewed as an assumption of statistical independence between the crossing at t and the prior crossing at η. Substituting N^+ into equation (23) and noting that

$$\int_0^\infty g_{\Gamma_1}(\dot{y}/\eta)\, d\dot{y} = 1$$

results in the equation

$$N^+(t/x,\dot{x},0) = f_{\Gamma_1}(x,\dot{x},t) + \int_0^t N^+ \; f_{\Gamma_1}(x,\dot{x},\eta)\, d\eta \qquad (24)$$

For convenience the independent crossing model is called approximation A, and the solution resulting from this approximation is denoted $f^A_{\Gamma_1}(x,\dot{x},t)$.

A simple connection with elementary work on the first passage problem can be obtained from approximation A. If $N^+(t/x,\dot{x},0)$ is also approximated by N^+ equation (24) becomes

$$N^+ = f_{\Gamma_1}(x,\dot{x},t) + \int_0^t N^+ f_{\Gamma_1}(x,\dot{x},\eta)\, d\eta \qquad (25)$$

The solution to this equation is

$$f_{\Gamma_1}(x,\dot{x},t) = N^+ \exp\,(-N^+\, t) \qquad (26)$$

This solution is well known. It has been obtained in the literature on first passage problems from a variety of procedures [3, p. 70], [22, p. 108], [16, p. 105], etc. All of these are based on assumptions that are equivalent to those given above.

First Passage Time and Velocity Independence

In Approximation A the difficulty of not knowing an expression for $g_{\Gamma_1}(\dot{y}/\eta)$ was avoided by assuming independence between crossings. The approximation that is considered here is obtained by assuming independence of the time and velocity random variables $\tau(x,\dot{x})$ and $\dot{X}(\tau(x,\dot{x}))$. This assumption only implies that $g_{\Gamma_1}(\dot{y}/\eta)$ is not a function of η. Thus, it does not determine a specific form for $g_{\Gamma_1}(\dot{y}/\eta)$. However, it is possible to choose a form which is to some extent in accordance with the mechanism that governs the process being considered in this research.

The conditional density function of the random variable for the velocity of a positive crossing of the level b, given that a positive crossing has occurred, can be expressed using S. O. Rice's formula [23, pp. 1216 and 1217]. It is $\dot{y}\, p(b,\dot{y})/N^+$. Every positive crossing of the level b is a first passage for some time and initial state. Thus, if $\dot{X}(\tau(x,\dot{x}))$ is assumed independent of $\tau(x,\dot{x})$, there is little reason for differentiating between first passages and arbitrary positive crossings. Hence, the density

$$g^B_{\Gamma_1}(\dot{y}/\eta) = \frac{\dot{y}\, p(b,\dot{y})}{N^+}$$

is assumed for $g_{\Gamma_1}(\dot{y}/\eta)$. This approximation is called approximation B and the solution for the first passage time density function which results from this approximation is denoted $f^B_{\Gamma_1}(x,\dot{x},t)$.

Before proceeding it should be pointed out that by substituting $g^B_{\Gamma_1}(\dot{y}/\eta)$ into equation (23) it is possible to derive the integral equation on which J. R. Rice based his study of the first passage problem [15, p. 25]. This derivation and a discussion of J. R. Rice's work is given in Appendix D.

Initial Point Approximation

In a sense this approximation is a refinement of the previous approximation. In $g^B_{\Gamma_1}(\dot{y}/\eta)$ all the sample paths of the process which cross at time t with positive slope are included. In the problem being studied here only sample functions which start from the point $(x,\dot{x})$ are considered. This fact can be incorporated in an approximation for $g_{\Gamma_1}(\dot{y}/\eta)$ by using the density function $p(b,\dot{y},\eta/x,\dot{x},0)$ in place of $p(b,\dot{y})$.

This approximation is called approximation C, and the assumed density for the first passage time velocities is

$$g^C_{\Gamma_1}(\dot{y}/\eta) = \frac{\dot{y}\, p(b,\dot{y},\eta/x,\dot{x},0)}{N^+(\eta/x,\dot{x},0)}$$

The solution for the first passage time density resulting from this approximation is denoted by $f^C_{\Gamma_1}(x,\dot{x},t)$.

Independence of $X(\tau(x,\dot{x}))$ and $\tau(x,\dot{x})$ is not assumed in approximation C. What is assumed is that first passages and arbitrary crossings of the level b, for sample functions starting from $(x,\dot{x})$, have the same velocity distribution.

Evaluation of the Kernel Approximations

In this section the relationships between the kernel approximations are discussed in terms of the sizes of the time η and the time difference $t - \eta$. Also some conditions are developed which the conditional density function $g_{\Gamma_1}(\dot{y}/\eta)$ must satisfy. These conditions are applied to $g^B_{\Gamma_1}(\dot{y}/\eta)$ and $g^C_{\Gamma_1}(\dot{y}/\eta)$.

All the approximations for $g_{\Gamma_1}(\dot{y}/\eta)$ are greater than or equal to zero and when integrated yield one. The approximations suggested the following kernels for use in equation (23):

$$\text{(A)}\quad k^A\,(t,\eta) = N^+$$

$$\text{(B)}\quad k^B\,(t,\eta) = \int_0^\infty N^+(t/b,\dot{y},\eta)\,\frac{\dot{y}\,p(b,\dot{y})}{N^+}\,d\dot{y}$$

$$\text{(C)}\quad k^C\,(t,\eta) = \int_0^\infty N^+(t/b,\dot{y},\eta)\,\frac{\dot{y}\,p(b,\dot{y},\eta/x,\dot{x},0)}{(\eta/x,\dot{x},0)}\,d\dot{y}$$

The similarities between these kernels for large times and large time differences are as follows:

1. In the limit as $t - \eta$ approaches infinity, both $k^B\,(t,\eta)$ and $k^C\,(t,\eta)$ approach $k^A\,(t,\eta)$.

2. If the limit is taken as t tends to infinity with $t - \eta$ fixed, $k^C\,(t,\eta)$ approaches $k^B\,(t,\eta)$.

The differences between the kernel approximations may be discussed in terms of sample path continuity.[14] Suppose that a sample path which has a positive crossing of the level b at time η is considered and the likelihood of a second positive crossing within a dt interval of a time t (after η) is examined. If the time difference $t - \eta$ is small, sample path continuity would preclude the movement through the position-velocity space that is necessary to make the crossing at time t possible. This is reflected in the fact that

$$\lim_{\eta \to t^-} N^+(t/b,\dot{y},\eta) = 0$$

Approximation A is indifferent to this fact because it is not a function of t and η. Thus it assigns the same likelihood to the crossing at t independent of the time difference $t - \eta$. On the other hand, the sample continuity property is reflected in approximations B and C by their inclusion of $N^+(t/b,\dot{y},\eta)$. The difference between approximations B and C could be interpreted using sample continuity also. However, this difference will become apparent when the consitions that are derived in the following paragraphs for $g_{\Gamma_1}(\dot{y}/\eta)$ are applied to $g^B_{\Gamma_1}(\dot{y}/\eta)$ and $g^C_{\Gamma_1}(\dot{y}/\eta)$.

A bound on the joint density function $g_{\Gamma_1}(\eta,\dot{y})$ yields one condition. The first passages that occur in a time-velocity increment $d\eta d\dot{y}$ are also positive crossings. The flux of positive crossings

14. The fact that the sample functions of the processes considered in the present work are continuous in position and velocity is discussed in Appendix B.

through that increment is given by $\dot{y}\ p(b,\dot{y},\eta/x,\dot{x},0)\ d\eta d\dot{y}$. Thus

$$g_{\Gamma_1}(\eta,\dot{y}) \le \dot{y}\ p(b,\dot{y},\eta/x,\dot{x},0)$$

In terms of $g_{\Gamma_1}(\dot{y}/\eta)$ and $f_{\Gamma_1}(x,\dot{x},\eta)$ this gives

$$g_{\Gamma_1}(\dot{y}/\eta)\ f_{\Gamma_1}(x,\dot{x},\eta) \le \dot{y}\ p(b,\dot{y},\eta/x,\dot{x},0) \tag{27}$$

or

$$g_{\Gamma_1}(\dot{y}/\eta) \le \frac{\dot{y}\ p(b,\dot{y},\eta/x,\dot{x},0)}{f_{\Gamma_1}(x,\dot{x},\eta)} \tag{28}$$

Thus $g_{\Gamma_1}(\dot{y}/\eta)$ must satisfy the limit conditions

$$g_{\Gamma_1}(\dot{y}/\eta) = O(\dot{y}) \quad \text{as } \dot{y} \to 0$$

and

$$g_{\Gamma_1}(\dot{y}/\eta) = O(\dot{y}\ \exp(-\dot{y}^2/2\dot{\sigma}_\eta^{\ 2})) \quad \text{as } \dot{y} \to \infty$$

The second limit results from the fact that

$$p(b,\dot{y},\eta/x,\dot{x},0) = O(\exp(-\dot{y}^2/2\dot{\sigma}_\eta^2)) \quad \text{as } \dot{y} \to \infty$$

Both $g^B_{\Gamma_1}(\dot{y}/\eta)$ and $g^C_{\Gamma_1}(\dot{y}/\eta)$ satisfy this condition.

A second condition results from the fact that as η tends to zero the ratio of first passages to positive crossings approaches unity; i.e.,

$$\lim_{\eta \to 0} \frac{f_{\Gamma_1}(x,\dot{x},\eta)}{N^+(\eta/x,\dot{x},0)} = 1$$

This fact and inequality (28) indicate that $g_{\Gamma_1}(\dot{y}/\eta)$ must be asymptotic to

$$\frac{\dot{y}\ p(b,\dot{y},\eta/x,\dot{x},0)}{N^+(\eta/x,\dot{x},0)}$$

as η tends to zero. This is $g^C_{\Gamma_1}(\dot{y}/\eta)$. An examination of the quantity

$$\frac{\dot{y}\ p(b,\dot{y},\eta/x,\dot{x},0)}{N^+(\eta/x,\dot{x},0)}$$

will yield the fact that it approaches a delta type density (i.e., all the probability mass becomes concentrated at a single value) as η tends to zero and the mean value of this probability tends to infinity. Thus any approximation of $g_{\Gamma_1}(\dot{y}/\eta)$ which is not a function of time (such as $g^B_{\Gamma_1}(\dot{y}/\eta)$) cannot satisfy this condition. Therefore, only $g^C_{\Gamma_1}(\dot{y}/\eta)$ satisfies it.

It is possible to compare approximation C with the S. O. Rice exclusion series. For the region Γ_1 the exclusion series is

$$f_{\Gamma_1}(x,\dot{x},t) =$$

$$N^{+}(t/x,\dot{x},0) - \int_0^t N^{++}(s,t/x,\dot{x},0)\,ds + \int_0^t\int_0^s N^{+++}(r,s,t/x,\dot{x},0)\,drds$$

$$- \int_0^t\int_0^s\int_0^r N^{++++}(q,r,s,t/x,\dot{x},0)\,dqdrds + \ldots$$

where $N^{+}(t/x,\dot{x},0)$ has already been defined and $N^{++}(s,t/x,\dot{x},0)$, $N^{+++}(r,s,t/x,\dot{x},0)$ etc. are defined by

(a) $N^{++}(s,t/x,\dot{x},0)\,dsdt$ is the probability of having a positive crossings in the intervals $(s,s+ds]$ and $(t,t+dt]$, given that $(X(0),\dot{X}(0)) = (x,\dot{x})$. It can be expressed by the equation

$$N^{++}(s,t/x,\dot{x},0) = \int_0^\infty\int \dot{y}\,\dot{z}\,p(b,\dot{z},t/b,\dot{y},s)\,p(b,\dot{y},s/x,\dot{x},0)\,d\dot{y}d\dot{z}$$

(b) $N^{+++}(r,s,t/x,\dot{x},0)\,drdsdt$ is the probability of having positive crossings in the intervals $(r, r+dr]$, $(s, s+ds]$, and $(t, t+dt]$, given that $(X(0),\dot{X}(0)) = (x,\dot{x})$. It can be expressed by the equation

$$N^{+++}(r,s,t/x,\dot{x},0) =$$

$$= \int \int_0^\infty \int \dot{w}\ \dot{y}\ \dot{z}\ p(b,\dot{z},t/b,\dot{y},s)\ p(b,\dot{y},s/b,\dot{w},r)\ p(b,\dot{w},r/x,\dot{x},0)\ d\dot{w}d\dot{y}d\dot{z}$$

The higher order crossing probabilities are defined in a similar manner. The exclusion series is an exact expression for $f_{\Gamma_1}(x,\dot{x},t)$; however, the numerical evaluation of all but first few terms is prohibitive [16].

The insertion of approximation C into equation (23) yields

$$N^+(t/x,\dot{x},0) = f^C{}_{\Gamma_1}(x,\dot{x},t) + \int_0^t k^C\ (t,\eta)\ f^C{}_{\ 1}(x,\dot{x},\eta)\ d\eta \qquad (29)$$

where

$$k^C\ (t,\eta) = \frac{\int_0^\infty N^+(t/b,\dot{y},\eta)\ \dot{y}\ p(b,\dot{y},\eta/x,\dot{x},0)\ d\dot{y}}{N^+\ (\eta/x,\dot{x},0)}$$

Equation (29) is a Volterra equation of the second kind.

A Neumann-Liouville series can be generated for $f^C{}_{\Gamma_1}(x,\dot{x},t)$ from equation (29). It is

$$f^{C}_{\Gamma_1}(x,\dot{x},t) =$$

$$N^{+}(t/x,\dot{x},0) - \int_0^t k^{C}(t,s)\, N^{+}(s/x,\dot{x},0)\, ds +$$

$$+ \int_0^t \int_0^s k^{C}(t,s)\, k^{C}(s,r)\, N^{+}(r/x,\dot{x},0)\, dr ds -$$

$$+ \int_0^t \int_0^s \int_0^r k^{C}(t,s)\, k^{C}(s,r)\, k^{C}(r,q)\, N^{+}(q/x,\dot{x},0)\, dq dr ds$$

$$+ \ . \ . \ .$$

The integrands in this series can be expressed as follows:

(a) $k^{C}(t,s)\, N^{+}(s/x,\dot{x},0) =$

$$= \frac{\int_0^\infty\!\!\int \dot{y}\ \dot{z}\ p(b,\dot{z},t/b,\dot{y},s)\ p(b,\dot{y},s/x,\dot{x},0)\ d\dot{y}d\dot{z}\ N^{+}(s/x,\dot{x},0)}{N^{+}(s/x,\dot{x},0)}$$

$$= \int_0^\infty\!\!\int \dot{y}\ \dot{z}\ p(b,\dot{z},t/b,\dot{y},s)\ p(b,\dot{y},s/x,\dot{x},0)\ d\dot{y}d\dot{z}$$

(b) $\overset{C}{k}(t,s)\,\overset{C}{k}(s,r)\,N^{+}(r/x,\dot{x},0) =$

$$= \left\{ \begin{array}{l} \dfrac{\int_0^\infty\int \dot{y}\ \dot{z}\ p(b,\dot{z},t/b,\dot{y},s)\ p(b,\dot{y},s/x,\dot{x},0)\ d\dot{y}d\dot{z}}{N^{+}(s/x,\dot{x},0)} \times \\ \\ \times \dfrac{\int_0^\infty\int \dot{y}\ \dot{z}\ p(b,\dot{z},s/b,\dot{y},r)\ p(b,\dot{y},r/x,\dot{x},0)\ d\dot{y}d\dot{z}\ N^{+}(r/x,\dot{x},0)}{N^{+}(r/x,\dot{x},0)} \end{array} \right.$$

$$= \left\{ \begin{array}{l} \dfrac{\int_0^\infty\int \dot{y}\ \dot{z}\ p(b,\dot{z},t/b,\dot{y},s)\ \ p(b,\dot{y},s/x,\dot{x},0)\ d\dot{y}d\dot{z}}{N^{+}(s/x,\dot{x},0)} \times \\ \\ \times \int_0^\infty\int \dot{y}\ \dot{z}\ p(b,\dot{z},s/b,\dot{y},r)\ p(b,\dot{y},r/x,\dot{x},0)\ d\dot{y}d\dot{z} \end{array} \right.$$

etc.

From (a) above it is obvious that the first two terms in the Neumann-Liouville series agree with the first two terms in the exclusion series. The higher order terms would agree if the expressions for the multiple crossing probabilities could be decomposed into expressions involving pairs of crossings. Of course only sample paths that start from $(x,\dot{x})$ are to be considered.

CHAPTER IV

NUMERICAL RESULTS

In this chapter some numerical estimates of the first passage time density that result from approximation C suggested in Chapter III are presented. These results are compared with other methods of solution that have been suggested by K. L. Chandiramani [14], Franklin and Rodemich [20], and R. G. Cook [24].

The first passage problem that has been considered up to this point in the present work may be called the fixed start problem because in this problem only sample paths which start from the specified initial point $(x,\dot{x})$ are considered. Some of the previous work deals with a different type of first passage problem which may be called the stationary start problem. This problem may be stated as follows: consider the sample paths that start from an initial randomly located point $(X(0),\dot{X}(0))$ that is distributed in Γ_1 according to the first order density $p(x,\dot{x})$. Find the probability density function for the first passage from Γ_1. Numerical results will also be presented in the present chapter for this problem.

When dealing with the harmonic oscillator it is convenient to use the following dimensionless parameters:

(a) $\psi = \omega_0 t$ for time,

(b) $\rho = b/\sigma_\infty$ for the boundary level, and

(c) $\zeta = \beta/2\omega_0$ for the damping factor.

Except for one example where β and ω_0 are zero, the numerical results are presented in terms of these parameters. This will simplify the comparison with other work.

The Fixed Start Problem

Consider equation (29) of Chapter III. It is possible to obtain a numerical solution to this equation by using a numerical quadrature to generate an equivalent system of linear simultaneous equations [25, Sec. 12.8]. In the present work the trapezoidal rule is used as the numerical quadrature, and the form of the resulting linear equations is

$$N^+ (t_i/x,\dot{x},0) = f^C_{\Gamma_1}(x,\dot{x},t_i) + \sum_{j=1}^{i-1} k^C(t_i,t_j)\, f^C_{\Gamma_1}(x,\dot{x},t_j)\, \Delta t +$$

$$+ \frac{1}{2} k^C (t_i,0)\, f^C_{\Gamma_1}(x,\dot{x},0)\, \Delta t + \frac{1}{2} k^C (t_i,t_i)\, f^C_{\Gamma_1}(x,\dot{x},t_i)\, \Delta t \qquad (30)$$

Δt is a fixed time increment and $t_i - t_{i-1} = \Delta t$. The last two terms on the right in equation (30) are zero because $f^C_{\Gamma_1}(x,\dot{x},0) = 0$ and $k^C(t_i,t_i) = 0$. Equation (30) yields the linear system

$$\begin{bmatrix} 1 & & & \bigcirc \\ k^C(t_2,t_1)\,\Delta t & 1 & & \\ k^C(t_3,t_1)\Delta t & k(t_3,t_2)\Delta t & 1 & \\ \vdots & & \ddots & \\ k^C(t_n,t_1)\,\Delta t & \cdots & \cdots & 1 \\ \vdots & & & \ddots \end{bmatrix} \times \begin{bmatrix} f^C_{\Gamma_1}(x,\dot{x},t_1) \\ f^C_{\Gamma_1}(x,\dot{x},t_2) \\ f^C_{\Gamma_1}(x,\dot{x},t_3) \\ \vdots \\ f^C_{\Gamma_1}(x,\dot{x},t_n) \\ \vdots \end{bmatrix} = \begin{bmatrix} N^+(t_1/x,\dot{x},0) \\ N^+(t_2/x,\dot{x},0) \\ N^+(t_3/x,\dot{x},0) \\ \vdots \\ N^+(t_n/x,\dot{x},0) \\ \vdots \end{bmatrix}$$

This system can be solved by successive substitution. In order to accomplish this solution on a digital computer the numerical generation of $k^C(t_i,t_j)$ and $N^+(t_i/x,\dot{x},0)$ is required. $N^+(t_i/x,\dot{x},0)$ is given by

$$N^+(t_i/x,\dot{x},0) = \int_0^\infty \dot{y}\; p(b,\dot{y},t_i/x,\dot{x},0)\; d\dot{y}$$

This integral may be evaluated in closed form. The integral in the numerator of the kernel may be written as

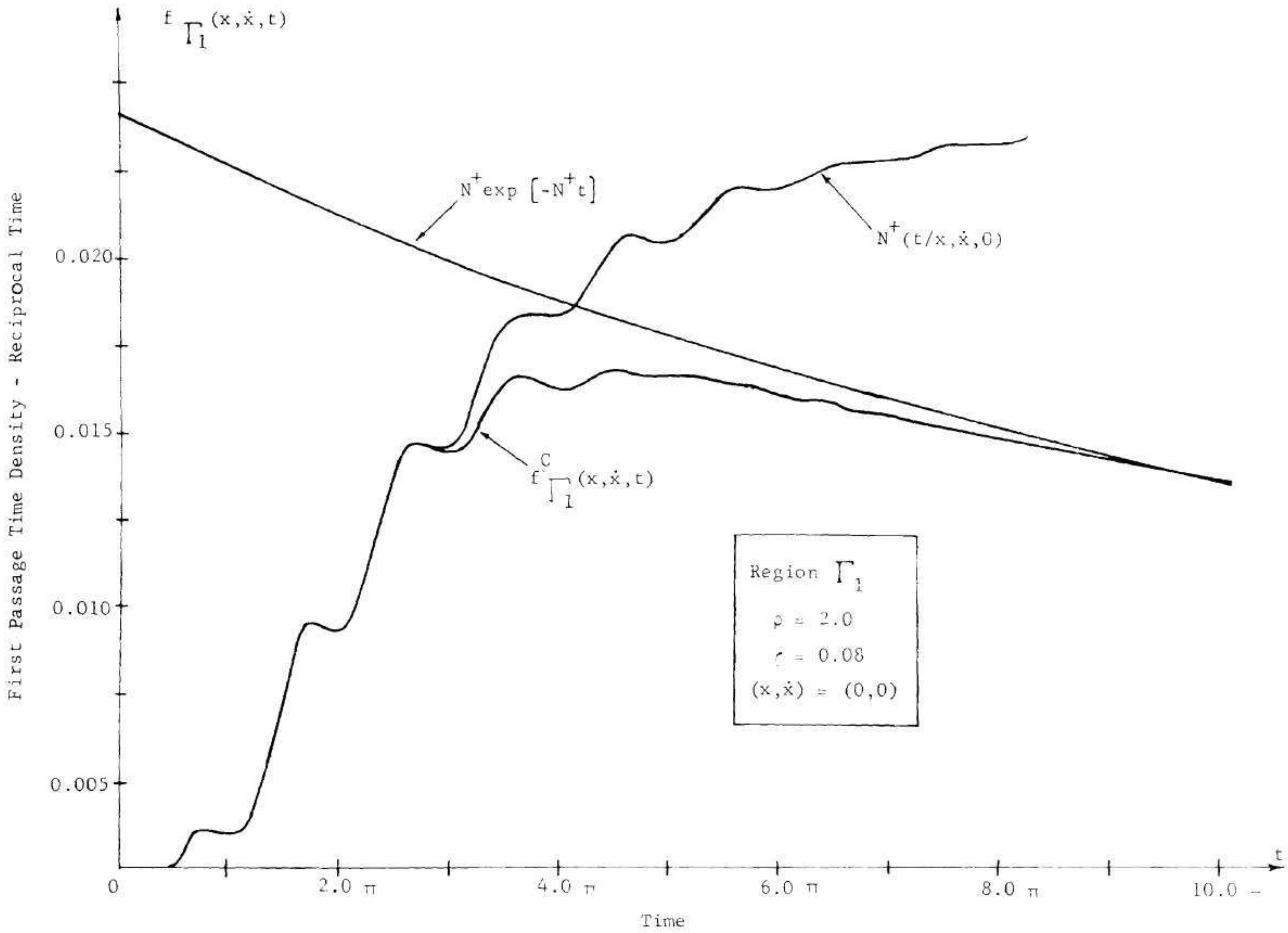

Figure 2. First Passage Time Probability Densities.

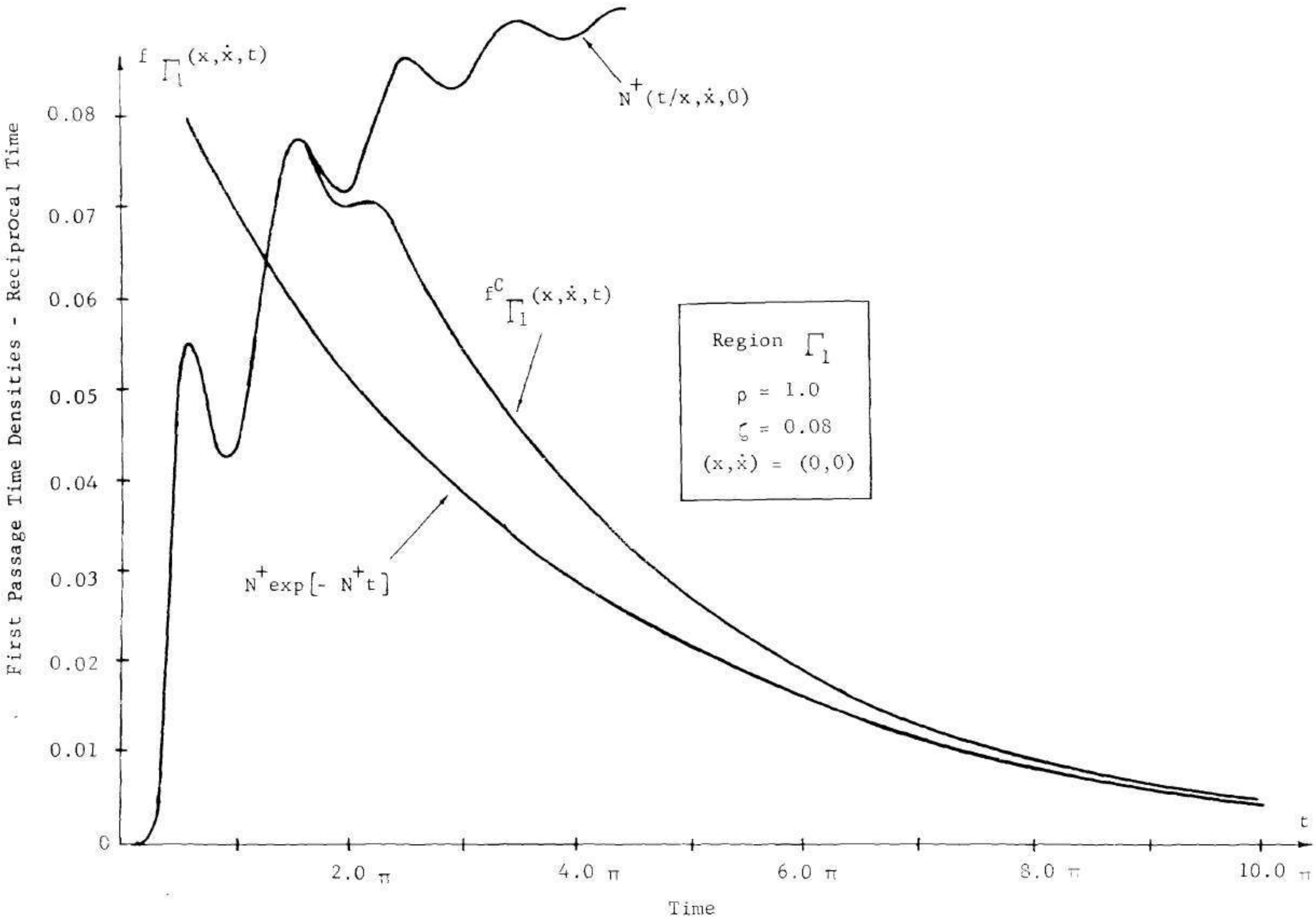

Figure 3. First Passage Time Probability Densities.

$$\int_0^{\infty} N^+ (t_i/b,\dot{y},t_j)\ \dot{y}\ p(b,\dot{y},t_j/x,\dot{x},0)\ d\dot{y} =$$

$$\int_0^{\infty}\int_0^{\infty} \dot{y}\ \dot{z}\ p(b,\dot{z},t_i/b,\dot{y},t_j)\ \ p(b,\dot{y},t_j/x,\dot{x},0)\ d\dot{y}\ d\dot{z}$$

A closed form expression for this integral does not appear to be available. Because $p(y,\dot{y},s/x,\dot{x},r)$ is Gaussian some simplification is possible. By conversion to polar coordinates the above integral may be reduced to a single integral over a finite range which may be evaluated by any one of a great many methods of numerical integration. The method of Gaussian integration is used in this work [26, p. 887].

Calculations have been carried out for $f^{C}_{\Gamma_1}(x,\dot{x},t)$ for several values of ζ and ρ. The results of these calculations appear in Figures 2 and 3. Also, $N^+(t/x,\dot{x},0)$, which is an upper bound for $f_{\Gamma_1}(x,\dot{x},t)$, and the negative exponential solution given in equation (26) are plotted in this figure.

Chandiramani [14] and Franklin and Rodemich [20] present results on the Γ_2 type region,

$$\Gamma_2 = \{(x,\dot{x})/-b < x \leq b,\ -\infty < \dot{x} < \infty\}$$

An equation which is analogous to equation (23) can be derived for this region. It is

$$N^{+}(t/x,\dot{x},0) + N_{-}(t/x,\dot{x},0) = f_{\Gamma_2}(x,\dot{x},t) + \int_0^t k(t,\eta)\, f_{\Gamma_2}(x,\dot{x},\eta)\, d\eta \qquad (31)$$

where

$$k(t,\eta) = \int_0^{\infty} (N^{+}(t/b,\dot{y},\eta) + N_{-}(t/b,\dot{y},\eta))\, g_{\Gamma_2}(\dot{y}/\eta)\, d\dot{y} + \int_{-\infty}^{0} (N^{+}(t/-b,\dot{y},\eta) + N_{-}(t/-b,\dot{y},\eta))\, g_{\Gamma_2}(\dot{y}/\eta)\, d\dot{y}$$

where $N_{-}(t/x,\dot{x},0)$ is the expected number of negative crossings per unit time of $-b$, given that the sample paths start from $(x,\dot{x})$. It is given by the equation

$$N_{-}(t/x,\dot{x},0) = -\int_{-\infty}^{0} \dot{y}\, p(-b,\dot{y},t/x,\dot{x},0)\, d\dot{y}$$

The density function $g_{\Gamma_2}(\dot{y}/\eta)$ is defined for both positive and negative values of $\dot{y}$ because first passages occur at $+b$ with positive velocities and at $-b$ with negative velocities. The form for $g_{\Gamma_2}(\dot{y}/\eta)$ which corresponds to approximation C is

$$g^{C}_{\Gamma_2}(\dot{y}/\eta) = \frac{h(\dot{y})\ \dot{y}\ p(b,\dot{y},\eta/x,\dot{x},0) - h(-\dot{y})\ \dot{y}\ p(-b,\dot{y},\eta/x,\dot{x},0)}{N^{+}(\eta/x,\dot{x},0) + N_{-}(\eta/x,\dot{x},0)} \qquad (32)$$

where $h(\dot{y})$ is the unit step function; i.e.,

$$h(\dot{y}) = \begin{cases} 1, & \dot{y} \geq 0 \\ 0, & \dot{y} < 0 \end{cases}$$

Substituting (32) into (31) yields

$$\left.\begin{array}{l} N^{+}(t/x,\dot{x},0) + \\ +N_{-}(t/x,\dot{x},0) \end{array}\right\} = f^{C}_{\neg 2}(x,\dot{x},t) + \int_{0}^{t} k^{C}_{2}(t,\eta)\, f^{C}_{\neg 2}(x,\dot{x},\eta)\, d\eta \qquad (33)$$

where

$$k^{C}(t,\eta) = \frac{\begin{array}{l} \int_{0}^{\infty} \left[N^{+}(t/b,\dot{y},\eta) + N_{-}(t/b,\dot{y},\eta)\right] \dot{y}\, p(b,\dot{y},\eta/x,\dot{x},0)\, d\dot{y} + \\ - \int_{-\infty}^{0} \left[N^{+}(t/-b,\dot{y},\eta) + N_{-}(t/-b,\dot{y},\eta)\right] \dot{y}\, p(b,\dot{y},\eta/x,\dot{x},0)\, d\dot{y} \end{array}}{N^{+}(\eta/x,\dot{x},0) + N_{-}(\eta/x,\dot{x},0)}$$

Equation (33) is a Volterra equation of the second kind, and it can be solved numerically by the method used to solve equation (29). Results for a few values of ζ and ρ appear in Figures 4, 5, 6, and 7. Also, $N^{+}(t/x,\dot{x},0) + N_{-}(t/x,\dot{x},0)$, which is an upper bound for $f_{\neg 2}(x,\dot{x},t)$, the negative exponential solution of equation (26), and the numerical estimate of $f_{\neg 2}(x,\dot{x},t)$ obtained by Chandiramani are plotted in these figures.

Franklin and Rodemich have presented an exact solution for the mean passage time $T_{\neg}(x,\dot{x})$ for the case where $\omega_0 = 0$, $\beta = 0$,

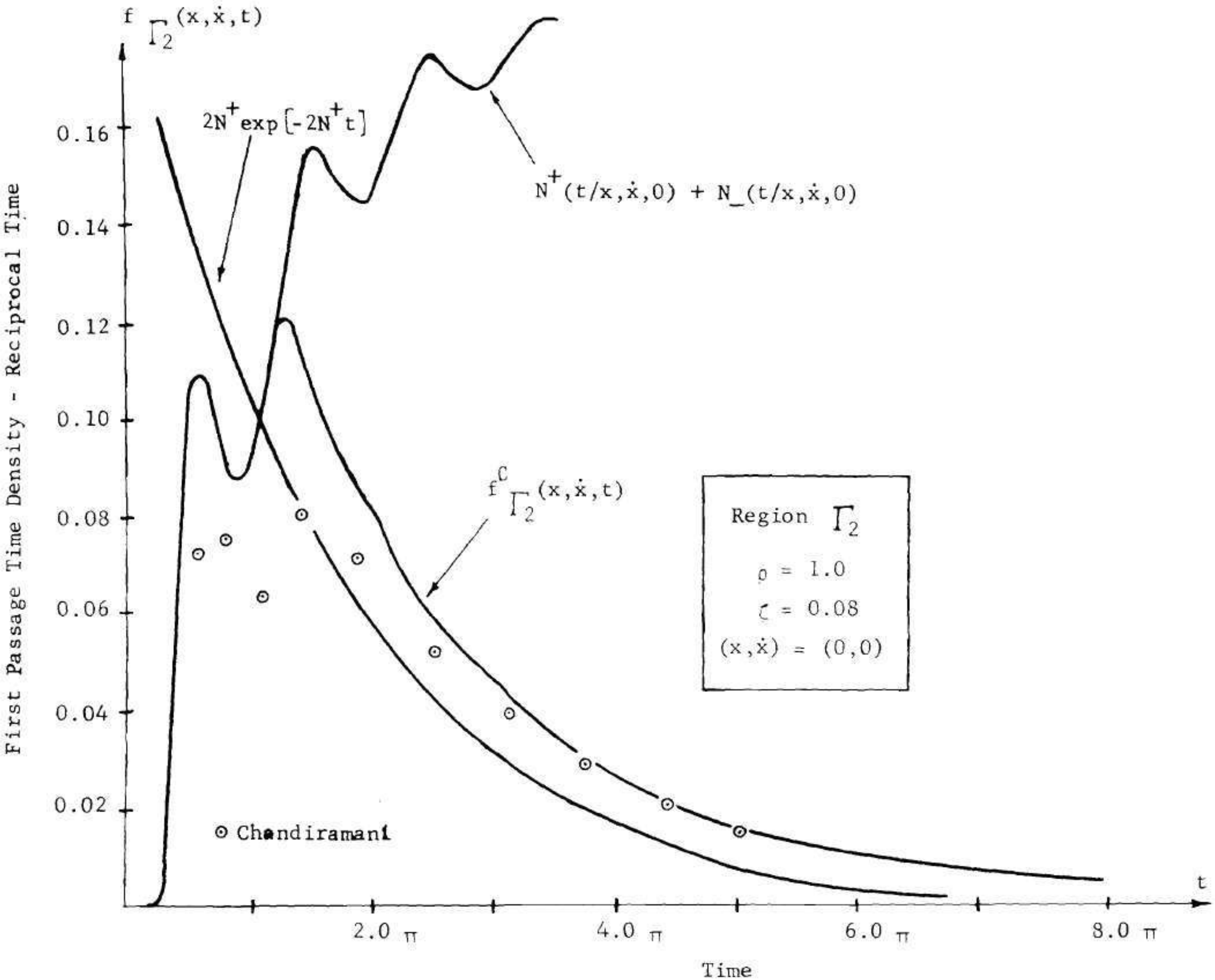

Figure 4. First Passage Time Probability Densities.

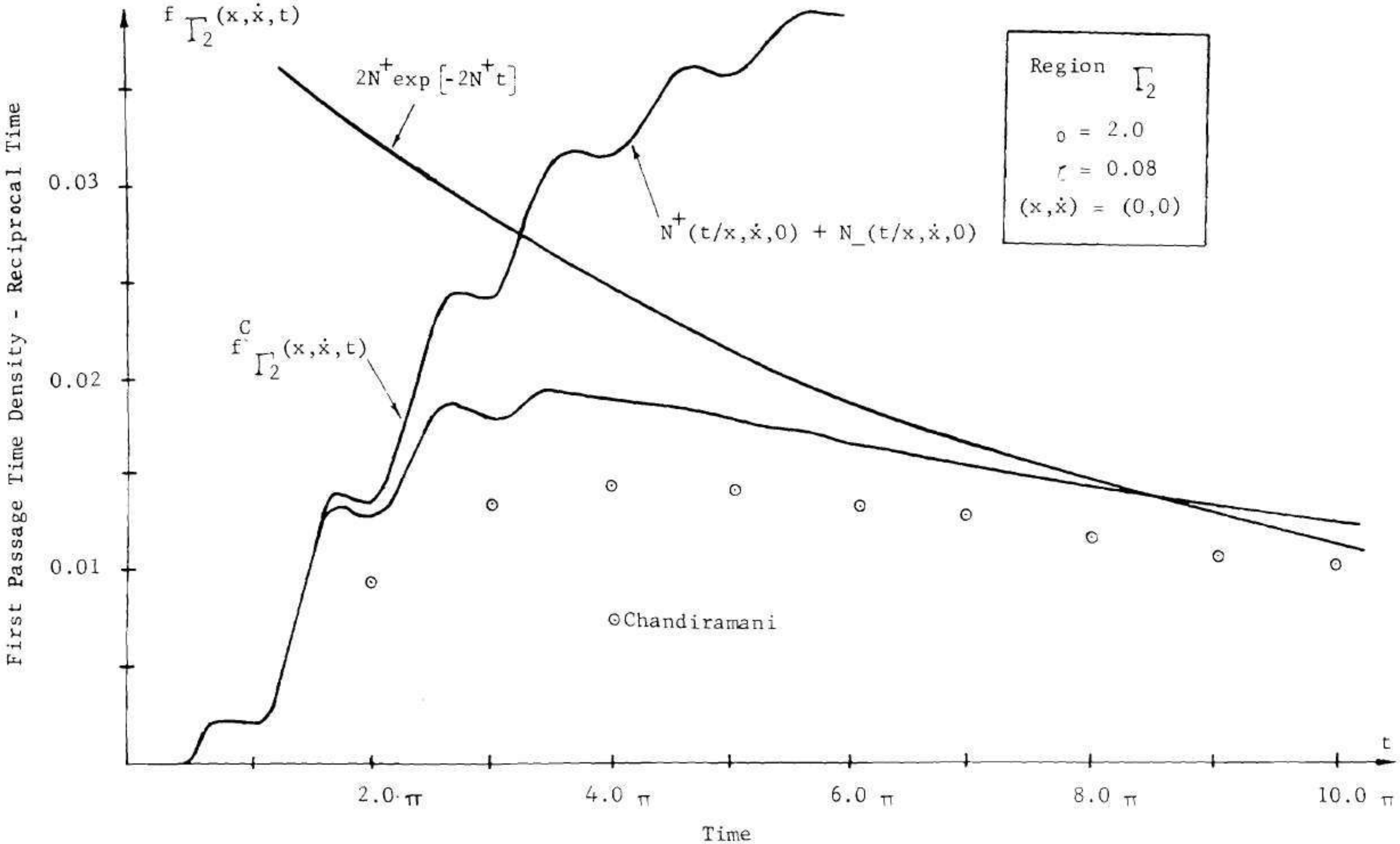

Figure 5. First Passage Time Probability Densities.

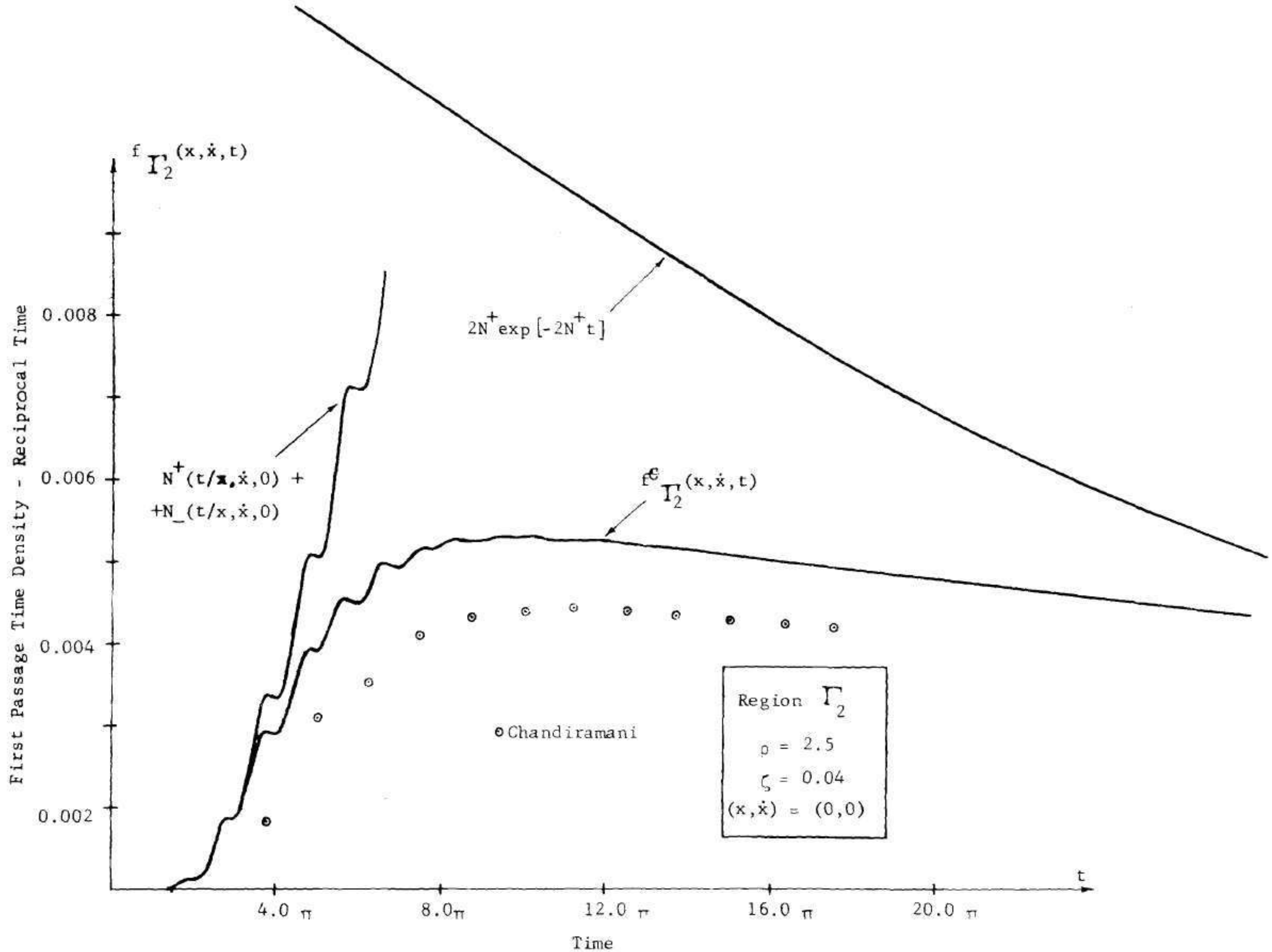

Figure 6. First Passage Time Probability Densities.

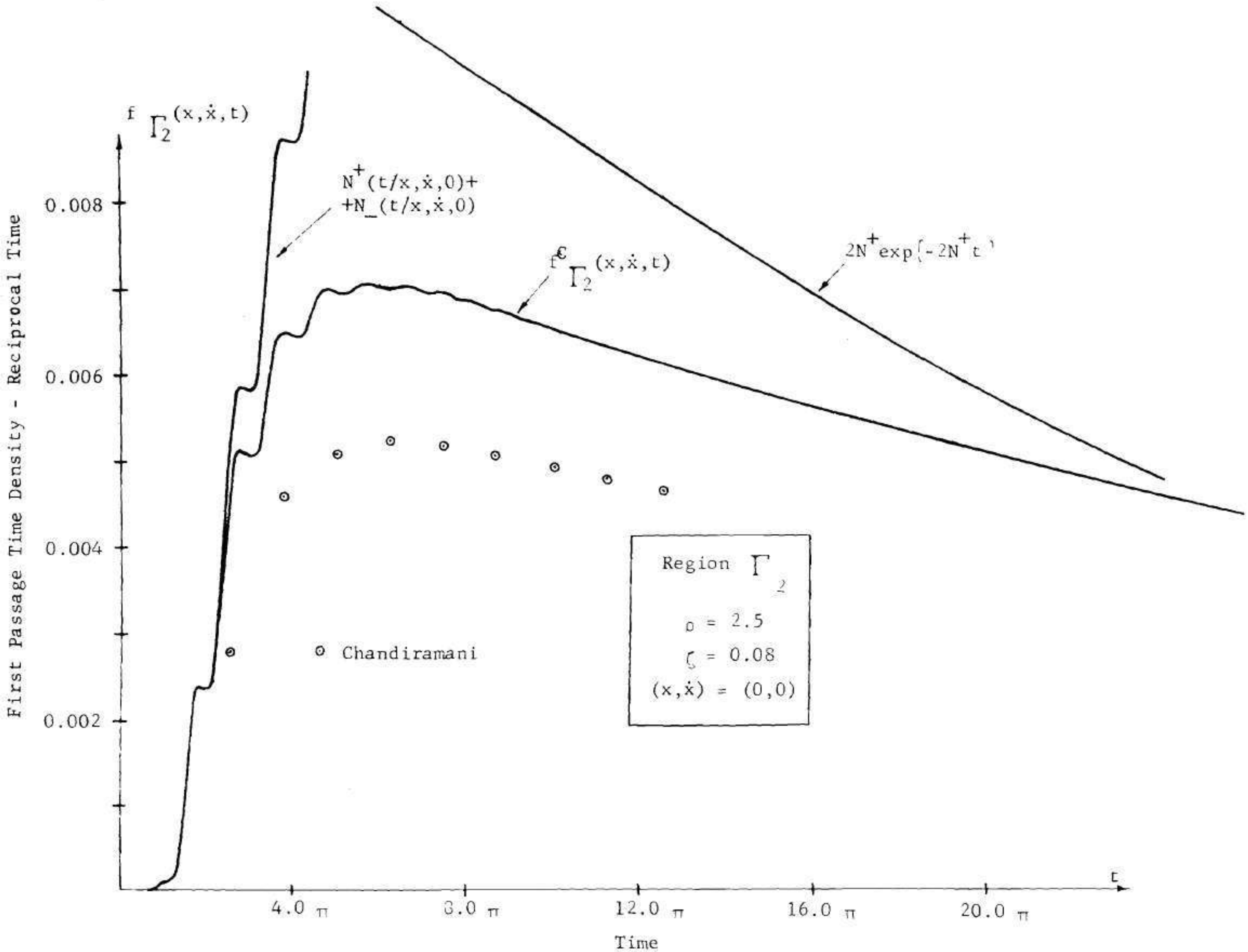

Figure 7. First Passage Time Probability Density.

$a = 1.0$, and $\Gamma_2 = \left\{(x,\dot{x})/1.0 < x \leq 1.0,\ -\infty < \dot{x} < \infty\right\}$.[15]

Their solution is

$$T_{\Gamma_2}(x,\dot{x}) = \begin{cases} -\dot{x}^2 + A_1\ (1-x)^{1/3}\ \dot{x}\ M\left(-\frac{1}{3}, \frac{4}{3}, -\frac{2}{9}\ \frac{\dot{x}^3}{1-x}\right) + \\ \quad + A_2\ \dot{x} \int_x^1 \exp\left(-\frac{2}{9}\ \frac{x^3}{\zeta - x}\right) \frac{v(\zeta)}{(\zeta - x)^{4/3}}\ d\zeta & \text{for } \dot{x} > 0 \\ v(x) \quad \text{for } \dot{x} = 0 \end{cases}$$

15. Their solution was obtained by solving the differential equation

$$\frac{a}{2}\ \frac{\partial^2\, T_{\Gamma_2}}{\partial \dot{x}^2} + \dot{x}\ \frac{\partial\, T_{\Gamma_2}}{\partial \dot{x}} = -1$$

which may be derived from the backward Fokker-Planck equation, with the boundary conditions

$$\lim_{x \to -b^+} T_{\Gamma_2}(x,\dot{x}) = 0,\ \dot{x} \leq 0$$

$$\lim_{x \to b^-} T_{\Gamma_2}(x,\dot{x}) = 0,\ \dot{x} > 0$$

where

(a) $v(x) = C\,(1 - x^2)^{\frac{1}{6}} \left[F(-\frac{1}{3},1,\frac{7}{6},\frac{1+x}{2}) + F(-\frac{1}{3},1,\frac{7}{6},\frac{1-x}{2})\right]$,

(b) $A = 6^{2/3}\ \Gamma(\frac{2}{3})\ /\ \Gamma(\frac{1}{3})$ (Γ is the gamma function.),

(c) $A = 2/6\ \Gamma(\frac{1}{3})$,

(d) $C = 3^{1/6}\ \ 2\ (\frac{1}{3})$,

(e) $F(a,b,c,z)$ is the hypergometric function [26, eq. 15.3.1], and

(f) $M(a,b,z)$ is the Kummer confluent hypergeometric function [26, eq. 13.1.2].

A comparison between this exact solution and the estimates obtained from approximation C is presented in Table 1.

Table 1. Comparison of Mean First Passage Times

	$T_{\Gamma_2}(x,\dot{x})$	
$(x,\dot{x})$	exact solution	approximation C
(0,0)	2.312	2.467
(0.2,0)	2.289	2.427
(0.4,0)	2.216	2.341
(0.6,0)	2.076	2.200
(0.8,0)	1.813	1.871

Stationary Start Problem

An integral equation for the stationary start problem can be derived from equation (23). The density function for the initial point random vector $(X(0), \dot{X}(0))$ is

$$p_{\Gamma_1}(x,\dot{x}) = \frac{p(x,\dot{x})}{\iint\limits_{\Gamma} p(x,\dot{x})\, dx d\dot{x}}$$

Let $h_{\Gamma_1}(\eta,\dot{y})$ be the joint density for the random variables τ and $\dot{X}(\tau)$ for the first passage time and first passage time velocity, respectively, given that the sample paths start from the randomly located initial point $(X(0), \dot{X}(0))$. And let $f_{\Gamma_1}(\eta)$ be the density for the random variable τ. Then

$$h_{\Gamma_1}(\eta,\dot{y}) = \iint\limits_{\Gamma_1} g_{\Gamma_1}(\eta,\dot{y}) \quad p_{\Gamma_1}(x,\dot{x}) \quad dx d\dot{x}$$

and

$$f_{\Gamma_1}(\eta) = \iint\limits_{\Gamma} f_{\Gamma}(x,\dot{x},\eta) \quad p_{\Gamma_1}(x,\dot{x})\, dx d\dot{x}$$

Equation (23) may be written as

$$N^{+}(t/x,\dot{x},0) = f_{\Gamma_1}(x,\dot{x},t) \quad + \int_0^t \int_0^\infty N^{+}(t/b,\dot{y},\eta)\; g_{\Gamma_1}(\eta,\dot{y})\; d\dot{y} d\eta$$

Multiplying by $p_{\Gamma_1}(x,\dot{x})$ and integrating over Γ_1 yields

$$N^+(t/\Gamma_1) = f_{\Gamma_1}(t) + \int_0^t \int_0^\infty N^+(t/b,\dot{y},\eta)\, h_{\Gamma_1}(\eta,\dot{y})\, d\dot{y}\, d\eta \tag{34}$$

where $N^+(t/\Gamma_1)$ is

$$N^+(t/\Gamma_1) = \iint_{\Gamma} N^+(t/x,\dot{x},0)\; p_{\Gamma_1}(x,\dot{x})\; dx\, d\dot{x}$$

The expression

$$\iint_{\Gamma_1} \dot{y}\; p(b,\dot{y},\eta/x,\dot{x},0)\; p_{\Gamma_1}(x,\dot{x})\; dx\, d\dot{x}\, d\dot{y}\, d\eta,\quad \dot{y} > 0$$

is the probability of having a positive crossing within a $d\eta d\dot{y}$ time-velocity interval of $(\eta,\dot{y})$, given that $(X(0),\ \dot{X}(0)) \in \Gamma_1$.

Thus the approximation

$$h^C_{\Gamma_1}(\dot{y}/\eta) = \iint_{\Gamma} \frac{\dot{y}\; p(b,\dot{y},\eta/x,\dot{x},0)\; p_{\Gamma_1}(x,\dot{x})\; dx\, d\dot{x}}{N^+(\eta/\Gamma_1)},\quad \dot{y} > 0$$

represents the same concept as approximation $g^C_{\Gamma_1}(\dot{y}/\eta)$ used in the fixed start problem. Using $h^C_{\Gamma_1}(\dot{y}/\eta)$, equation (34) can be reduced to the Volterra equation

$$N^+(t/\Gamma_1) = f^C_{\Gamma_1}(t) + \int_0^t k^C_3\,(t,\eta)\; f^C_{\Gamma_1}(t)\; d\eta \tag{35}$$

where

$$k^C_3(t,\eta) = \frac{\iint\limits_{\Gamma} \int\limits_0^{\infty} N^+(t/b,\dot{y},\eta)\ \dot{y}\ p(b,\dot{y},\eta/x,\dot{x},0)\ p_{\Gamma_1}(x,\dot{x})\ d\dot{y}\ dxd\dot{x}}{N^+(\eta/\Gamma_1)}$$

Equation (35) can be solved by the same method as equation (29). If the dimensionless ration ρ is large then $k^C_3(t,\eta)$ and $N\ (t/\Gamma_1)$ may be approximated by expressions which are much more easily computed. For ρ large, the majority of the probability mass is within Γ_1 at $t = 0$. (Table 2 gives the percentage of the total probability that is within Γ_1 for a stationary start for different values of ρ.) Thus it is appropriate to approximate the expression

$$\iint\limits_{\Gamma_1} \dot{y}\ p(b,\dot{y},\eta/x,\dot{x},0)\ p_{\Gamma_1}(x,\dot{x})\ dxd\dot{x}$$

by

$$\iint\limits_{-\infty}^{\infty} \dot{y}\ p(b,\dot{y},\eta/x,\dot{x},0)\ p(x,\dot{x})\ dxd\dot{x}$$

This integral is $\dot{y}\ p(b,\dot{y})$. Using this approximation in the expressions for $N^+\ (t/\Gamma_1)$ and $k_3^C(t,\eta)$ yields

$$N^+(t/\Gamma_1) \approx N^+$$

and

$$k_3^C(t,\eta) \approx \frac{\int\limits_0^{\infty} N^+(t/b,\dot{y},\eta)\ \dot{y}\ p(b,\dot{y})\ d\dot{y}}{N^+}$$

where $\approx$ means approximately equal. With these approximations results were obtained for several values of ζ and ρ. $f^C_{\Gamma_1}(t)$ for $\zeta = 0.01$

Table 2. Percentage of Probability Mass in Γ_1 at $t = 0$ for Different Values of ρ

ρ	percentage in Γ_1
1.0	84.13
1.5	93.32
2.0	97.72
3.0	99.87
3.5	99.97

and $\rho = 2.0$ is given in Figure 8. Also, the negative exponential solution, given in equation (26), and the numerical estimate which R. G. Cook obtained from a Monte Carlo study appear in this figure.

For high values of ρ the major portion of $f_{\Gamma_1}(t)$ is represented sufficiently well by an exponential curve of the form $A\alpha e^{-\alpha t}$ (A and α are determined by fitting the exponential to $f^C_{\Gamma_1}(t)$ for large t) that the mean first passage time may be estimated. Thus

$$E\left[T_{\Gamma_1}\right] = \int_0^\infty t\ f_{\Gamma_1}(t)\ dt \approx \int_0^\infty t\ A\alpha e^{-\alpha t} dt = \frac{A}{\alpha} \tag{36}$$

A comparison between estimates of the mean first passage time obtained from approximation C using the technique discussed above, $1/N^+$, which results from the negative exponential solution given in equation (26),

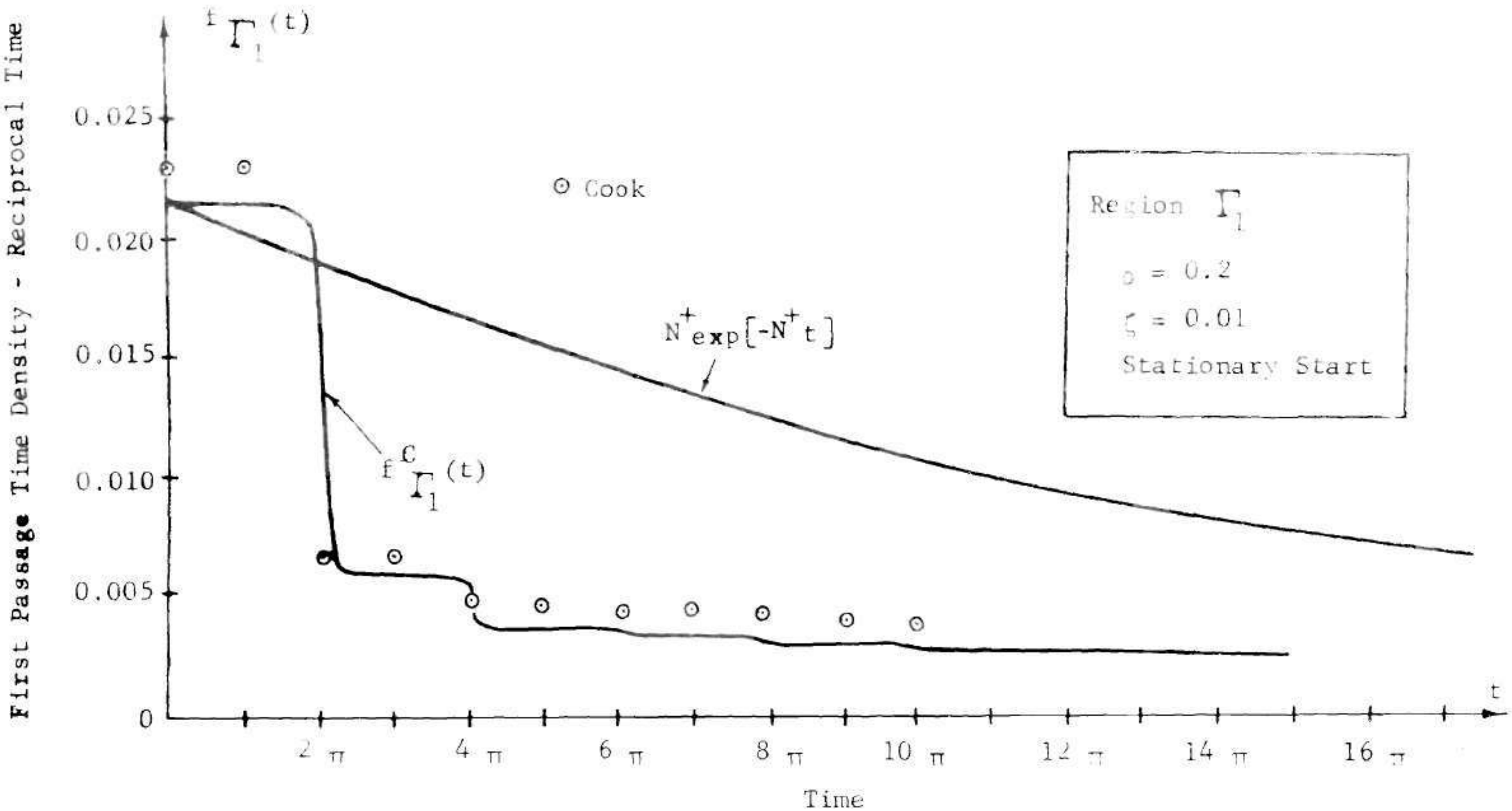

Figure 8. First Passage Time Probability Densities.

and the estimates obtained by R. G. Cook [27, p. 537] are presented in Table 3.

Table 3. Mean First Passage Time in Undamped Natural Periods.

ζ	ρ	$\frac{1}{N^+}$	$\frac{A}{\alpha}$ from C	R. G. Cook
0.04	2.0	7.38	12.5	12.6
0.04	2.5	22.77	40	39.8
0.04	3.0	90.42	137	146
0.04	3.5	457.1	664	676
0.01	2.0	7.38	35	31.7
0.01	2.5	22.77	105	97
0.01	3.0	90.42	347	338
0.01	3.5	457.1	1470	1510

Discussion of the Results

Agreement between the first passage time density estimates obtained using approximation C and the work of Chandiramani and Cook appears to be quite good. It is of interest to note that the estimates of the mean first passage times presented in Table 3 indicate that the negative exponential, $N^+e^{-N^+t}$, is a very conservative estimate when the damping is small. This fact was also noted by Chandiramani and Cook.

Chandiramani has expressed the opinion that the first passage time estimates that he obtained are lower bound estimates [14, p. 59]. Thus the fact that $f^{C}_{\Gamma_2}(x,\dot{x},t)$ on Figures 4, 5, 6, and 7 is above his estimate is not an unreasonable result. In fact, it can be shown that as long as $f^{C}_{\Gamma_2}(x,\dot{x},t)$ agrees with the upper bound on these figures, it is very close to the exact solution.[16]

In the stationary start problem, $f_{\Gamma_1}(t)$ should approach N^{+} as t tends to zero because initially all the crossings are first crossings. Thus the values presented in Figure 8 for Cook's estimate appear to be about eight per cent high. These values were obtained from a small graph [27, Fig. 10]. The scale on this graph may be slightly high. Thus better agreement could exist between approximation C and Cook's work than is indicated here.

16. The upper bound on these plots is a one term Rice exclusion series. A two term exclusion series would be a lower bound for the first passage time density. If one were plotted here, it would agree with $f^{C}_{\Gamma_2}(x,\dot{x},t)$ until the difference between $f^{C}_{\Gamma_2}(x,\dot{x},t)$ and the upper bound becomes large. Thus as long as these values are close the exact first passage time density should be close to them.

CHAPTER V

CONCLUDING REMARKS

The objective of this work has been the development and evaluation of a method for computing first passage probabilities. This has been accomplished with the numerical procedure given by equation (30) and the subsequent exploration of the damped harmonic oscillator that is presented in Chapter IV. A favorable comparison with results obtained by numerical simulation procedures has been noted in Chapter IV.

Two other comparsions are made with previous work on the first passage problem. In Chapter III the method that is developed here is compared with the S. O. Rice exclusion series. Also, it is illustrated in what sense the present analyais is an extension of the work of J. R. Rice. In appendix D the basic renewal equation of Rice's work is derived using approximation B of Chapter III.

One more important feature of the method presented here remains to be discussed. It is the extension to higher order systems. In order to illustrate this extension the system in Figure 9 and the region

$$\Gamma_1 = \left\{(x^1, \dot{x}^1, x^2, \dot{x}^2) \mid x^2 \leq b\right\}$$

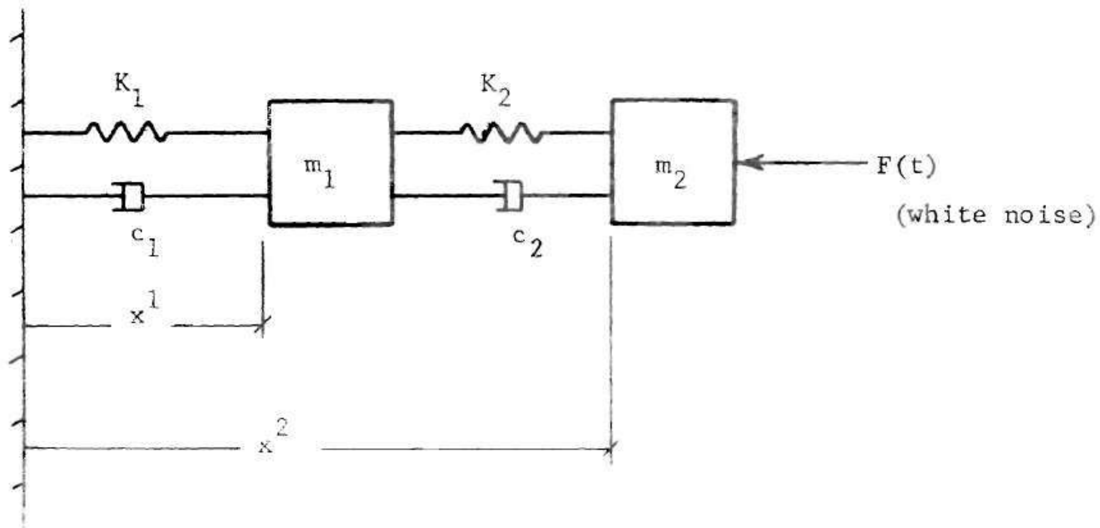

Figure 9. The Coupled Oscillator.

are considered. The response of this system is a Markov process on a four-dimensional position-velocity space. The probability densities $p(y^1,\dot{y}^1,y^2,\dot{y}^2,s/x^1,\dot{x}^1,x^2,\dot{x}^2,r)$ and $p(x^1,\dot{x}^1,x^2,\dot{x}^2)$ specify this process completely. An equation can be derived for this system which is exactly analogous to equation (23) of Chapter II. It is

$$N^+(t/x^1,\dot{x}^1,x^2,\dot{x}^2,0) = f_{\Gamma_1}(x^1,\dot{x}^1,x^2,\dot{x}^2,t) + \int_0^t \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_0^{\infty} N^+(t/y^1,\dot{y}^1,b^2,\dot{y}^2,\eta) \times$$

$$\times g_{\Gamma_1}(y^1,\dot{y}^1,\dot{y}^2/\eta) \quad f_{\Gamma_1}(x^1,\dot{x}^1,x^2,\dot{x}^2,\eta)\, d\dot{y}^2 dy^1 d\dot{y}^1 d\eta \tag{37}$$

Here $g_{\Gamma_1}(y^1,\dot{y}^1,\dot{y}^2/\eta)$ is the conditional density for the random vector $(X^1(\tau(x,\dot{x})),\ \dot{X}^1(\tau(x,\dot{x})),\ \dot{X}^2(\tau(x,\dot{x})))$, given that a first passage has occurred at $\tau = \eta$. $N^+(t/x^1,\dot{x}^1,x^2,\dot{x}^2,0)$ is the expected number of crossings of the level $x = b$, given that $(X^1(0),\ \dot{X}^1(0),\ X^2(0),\ \dot{X}^2(0)) = (x^1,\dot{x}^1,x^2,\dot{x}^2)$. $f_{\Gamma_1}(x^1,\dot{x}^1,x^2,\dot{x}^2,t)$ is the first passage time density. The approximation that corresponds to approximation C of Chapter III is

$$g^C_{\Gamma_1}(y^1,\dot{y}^1,\dot{y}^2/\eta) = \frac{\dot{y}^2\ p(\dot{y}^1,\dot{y}^1,b,\dot{y}^2,\eta/x^1,\dot{x}^1,x^2,\dot{x}^2,0)}{N\ (\eta/x^1,\dot{x}^1,x^2,\dot{x}^2,0)}$$

Substituting this into equation (37) yields

$$N^+(t/x^1,\dot{x}^1,x^2,\dot{x}^2,0) = f_{\Gamma_1}(x^1,\dot{x}^1,x^2,\dot{x}^2,t) +$$

$$+ \int_0^t k_4^C\,(t,\eta)\; f_{\Gamma_1}(x^1,\dot{x}^1,x^2,\dot{x}^2,\eta)\, d\eta$$

where $k_4^C\,(t,\eta)$ is

$$k_4^C\,(t,\eta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\infty} N^+(t/y^1,\dot{y}^1,b,\dot{y}^2,\eta)\; g^C_{\Gamma_1}(y^1,\dot{y}^1,\dot{y}^2/\eta)\; dy^1 d\dot{y}^1 d\dot{y}^2$$

The numerical solution to this equation requires the same amount of machine effort as equation (29).

APPENDICES

APPENDIX A

A STOCHASTIC MODEL

The purpose of this appendix is to give a definition of a stochastic process. Also, several basic concepts of stochastic analysis are discussed in conjunction with this definition.

Definition of a Stochastic Process

Physically, a stochastic process is a collection of functions of time $X_\omega(t)$ (ω indexes the members of the collection) that represent the possible outcomes of a physical experiment along with a method of assigning probabilities to events of interest. For example, the set of samples that have $X_\omega(t) \le x$ at time t. In order to insure that all events of interest will have a unique and consistence probability a rather precise mathematical definition is required. For a process on a Euclidean N-dimensional space, such a definition can be given in the following manner:

Definition: A stochastic process is a collection of random vectors $\{\bar{X}(t),\ t \in T\}$ on a probability space $(\Omega, \mathcal{A}, \mathcal{P})$. The indexing set T is a set of times, usually the interval $(-\infty,\infty)$. Ω is the set of vector-valued functions on an N-dimensional Euclidean space; i.e., Ω is $R_T^N = \prod_{t \in T} R_t^N$. $\mathcal{A}$ is the smallest σ-algebra over the leftopen right-closed vector intervals in R_T^N.[1A] The probability measure $\mathcal{P}$ is defined by the hierarchy of finite-dimensional distributions of the form

$$F\ (x_1^1, \ldots, x_1^N,\ t_1; \ldots; x_k^1, \ldots x_k^N, t_k)$$

or in vector notation

$$F\ (\bar{x}_1, t_1; \ldots; \bar{x}_k, t_k)$$

The relationship which accomplishes this is

$$F\ (x_1^1, \ldots, x_1^N, t_1; \ldots; x_k^1, \ldots, x_k^N,\ t_k) =$$
$$= \mathcal{P}\left(\left\{\omega / X_\omega^1\ (t_1) \leq x_1^1, \ldots, X_\omega^N\ (t_1) \leq x_1^N; \ldots; X_\omega^1\ (t_k) \leq x_k^1, \ldots, X_\omega^N (t_k) \leq x_k^N\right\}\right) \tag{1A}$$

Then the Laniell-Kolmogorov theorem guarantees the consistency of the probabilities [28, p. 30].

The members of Ω, denoted generically by ω, are the sample functions of the process. The random vectors in the collection $\{\bar{X}(t), t \in T\}$ can be defined by the mapping $\bar{X}_\omega = \omega$. The individual random vector $\bar{X}(t)$ maps Ω into R_t^N, and $\bar{X}_\omega(t)$ is the value of the function ω at time t (a point in R_t^N). Therefore $\bar{X}_\omega(t)$ could be considered to be a sample function of the process.

1A. A vector interval in R_t^N is $(\bar{a}_t, \bar{b}_t] = \prod_{i=1}^{N} (a_t^i, b_t^i]$ where $(a_t^i, b_t^i]$ is a component interval in a one-dimensional Euclidean space. A vector interval in R_T^N is a finite product $\prod_{j=1}^{k} (\bar{a}_{tj}, \bar{b}_{tj}]$.

Remarks Concerning the Definition of a Stochastic Process

A great deal of information may be extracted from the finite dimensional distribution functions. For example, for the velocity of a Brownian particle, it is possible to determine the probability of the sample paths that are in an interval (a_t, b_t) at time t. It is $F(a,t) - F(b,t)$. In fact the probability of any event that can be specified with a finite set of times can be expressed in terms of the finite-dimensional distributions.

Since much practical information can be obtained from the finite-dimensional distributions and because they play such a basic role in specifying Ω and $\mathcal{P}$ in a precise mathematical model, the following question appears quite appropriately. Other than the satisfaction of being able to specify a mathematically consistent model, why consider such an intricate definition that involves such a complex model?

The answer to this question is that a precise mathematical model forms a basis for the analysis of many important questions concerning the analytic properties of the sample functions of a stochastic process. For example, under the restrictions of regularity and separability,[2A] it is possible to discuss probabilities such as

(a) the probability of the set of continuous functions,

(b) the probability of the set of ten times differentiable functions,

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2A. Regularity and separability are discussed in detail in the next two sections of this appendix. It is important here to note that these restrictions are imposed only to circumvent mathematical difficulties, and they in no way hamper the analysis of a physical problem.

(c) the probability of the set of sample functions of a process on R^1 which are Riemann integrable over the time interval $[a,b]$,

(d) the probability of the set of sample functions of a process on R^1 for which the Riemann-Stieltjes integrals

$$\int_a^b f(t)\, dX_\omega(t)$$

exist for a continuous function f,

(e) the probability of the set of sample functions which start at $X_\omega(0) = x_0$ and remain in some region Γ over a time interval $(0,t)$, and

(f) the probability of the set of Riemann integrable sample functions of a process on R^1 for which

$$\int_a^b X_\omega(t)\, dt \le c$$

where c is a real number, given that $X(a) = x_a$ and $X(b) = x_b$.

Without a complex model for a stochastic process any discussion of such probabilities must appeal almost completely to intuition for the justification of mathematical operations.[3A]

Regularity

Often in the analysis of a stochastic process it is desirable to consider only the sample functions that pass through a given point $\bar{x}_1$ at a given time t_1 or the sample functions that pass through two points $\bar{x}_1$ at t_1 and $\bar{x}_2$ at t_2. (Such is the case in (e) and (f) in the preceding section.) For most physical problems the probability measure $\mathcal{P}$ is not concentrated at single elements (i.e., the probability that a sample function takes on a particular value is zero). Thus studying such a collection with the probability space $(\Omega, \mathcal{a}, \mathcal{P})$ would be meaningless, and a more appropriate probability space is needed.

Suppose the collection of sample paths which pass through $\bar{x}$ at $t = 0$ is considered, and an attempt at the construction of a suitable probability space is made. Specifying the sample space is no problem. It is the space of all real valued functions that pass through $\bar{x}$ at $t = 0$ $(\Omega(\bar{x},0) = \left\{\omega / \bar{x}_{\omega}(0) = \bar{x}\right\})$. The σ-algebra $\mathcal{a}(\bar{x},0)$ may be

3A. The application of intuition to stochastic analysis often has rather strange manifestations. For example, intuitively it might be expected that a function that represents the velocity of a Brownian particle is differentiable. In a stochastic model this means that the set of sample functions that are differentiable is a set of probability one. A cautious analysis of the widely accepted model for the velocity of a Brownian particle yields the fact that while the sample functions are continuous with probability one the probability of the set of differentiable sample functions is zero [28, pp. 255-260]. Thus analytic operations that depend on differentiability would be meaningless.

Intuitively, it would not be expected that a sample function representing a physical phenomenon would take on the values $\pm\infty$. Therefore, it might seem unnatural to include these values in a mathematical model. However, a model that does not allow these two values would not be separable [29, p. 68], and the operations listed in this section would be meaningless.

specified over $\Omega(\bar{x},0)$ in the same manner that α was specified over Ω. A logical choice for the specification of $\mathcal{P}(\cdot/\bar{x},0)$ [4A] over the sets in $\alpha(\bar{x},0)$ would be through the finite-dimensional conditional distribution functions of the form $F(\bar{x}_1,t_1;\ldots;\bar{x}_k,t_k/\bar{x},0)$ where $t_1,\ldots,t_k$ is any arbitrary set of k times.[5A] The Radon-Nikodym theorem [13, p. 132] determines each of these conditional distributions in such a manner that

$$F(\bar{x},0;\bar{x}_1,t_1;\ldots;\bar{x}_k,t_k) = \int_{(-\infty,\bar{x}_0]} F(\bar{x}_1,t_1;\ldots;\bar{x}_k,t_k/\bar{x},0)\,d_{\bar{x}}\,F(\bar{x},0) \quad (2A)$$ [6A]

This choice would yield the intuitively pleasing result that $\mathcal{P}$ could be thought of as the weighted sum (integral) of the $\mathcal{P}(\cdot/\bar{x},0)$ for each $\bar{x}$ in R^N.

Unfortunately there is a problem in this specification of $\mathcal{P}(\cdot/\bar{x},0)$. The Radon-Nikodym theorem does not specify the finite-dimensional distributions uniquely. It specifies each up to an exceptional set E of probability zero; i.e.,

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4A. If A is a set of functions in α then $\mathcal{P}(A/\bar{x},0)$ is the probability that A occurs given that $\bar{y}(0) = \bar{x}$. If A contains no ω such that $\bar{y}_\omega(0) = \bar{x}$, then $\mathcal{P}(A/\bar{x},0)$ is zero.

5A. This is the distribution function for the random vectors $\bar{y}(t_1),\ldots,\bar{y}(t_k)$, given $\bar{y}(0) = \bar{x}$.

6A. The integral here is the Lebesgue-Stieltjes integral over the set $(-\infty,\bar{x}_0]$ with respect to the measure induced by the distribution function $F(\bar{x},0)$.

$$\mathcal{P}\left(\left\{\omega / X_{\omega}(0) \in E\right\}\right) = \int_{E} d_{\bar{x}}\ F(\bar{x},0) = 0$$

Each conditional distribution might have a different exceptional set. Thus $\mathcal{P}(\cdot/\bar{x},0)$ could only be specified for $\bar{x}$ in the complement of the union of all the exceptional sets. This union is uncountable because non-denumerably many distribution functions $F(\bar{x}_1,t_1;\ldots;\bar{x}_k,t_k/\bar{x},0)$ are needed for the specification of $\mathcal{P}(\cdot/\bar{x},0)$. The uncountable union of $\bar{x}$ sets of probability zero could possibly be the whole space R^N. (R^N is the uncountable union of point sets.) If it is possible to determine the conditional distributions associated with each finite-dimensional distribution uniquely, the process is termed regular. For a regular process it is possible to construct probability spaces such as $(\Omega(\bar{x},0),\ a(\bar{x},0),\ \mathcal{P}(\bar{x},0))$ and consider subcollections of sample functions as stochastic processes in themselves.

Restricting consideration to regular processes seldom causes difficulties in the analysis of physical problems. In most instances the specification of the conditional distributions is known, and the finite-dimensional distributions are specified from these through integrations such as in equation (2A). The responses of lumped parameter models of structures to white noise are examples of this situation. Such processes are Markov processes on Euclidean spaces of one or more dimensions [4, p. 271], and two functions exist that specify them. The functions are the density function $p(\bar{x},r)$ for the random vector $\bar{X}(r)$

and $p(\bar{y},s/\bar{x},r)$ for the random vector $\bar{X}(s)$, given $\bar{X}(r) = \bar{x}$.[7A] These functions are defined for $-\infty<r<s<\infty$. The probability distribution function $F(\bar{x},r)$ and the conditional distribution function $F(\bar{y},s/\bar{x},r)$ can be obtained in the standard manner. They are

$$F(\bar{x},r) = \int_{(-\infty,\bar{x}]} p(\bar{z},r)\, d\bar{z}$$

and (3A)

$$F(\bar{y},s/\bar{x},r) = \int_{(-\infty,y]} p(\bar{z},s/\bar{x},r)\, d\bar{z}$$

The integrals are Lebesgue integrals with respect to N-dimensional Lebesgue measure (volume measure).

Any distribution $F(\bar{x}_1,t_1;\ldots;\bar{x}_k,t_k)$ in the hierarchy of distribution function that specify the process may be expressed using the Markov property. If $\infty < t_1 < \ldots < t_k < \infty$, the expression is

$$F(\bar{x}_1,t_1;\ldots;\bar{x}_k,t_k) =$$

(4A)

$$\int_{(-\infty,\bar{x}_1]\times\ldots\times(-\infty,\bar{x}]} p(\bar{z}_k,t_k/\bar{z}_{k-1},t_{k-1})\ldots p(\bar{z}_2,t_2/\bar{z}_1,t_1)\; p(\bar{z}_1,t_1)(d\bar{z}_1\times\ldots\times d\bar{z}_k)$$

7A. The determination of such functions is discussed in Appendix B.

The integration is with respect to k times N-dimensional Lebesgue measure. The conditional distributions $F(\bar{x}_1, t_1; \ldots; \bar{x}_k, t_k / \bar{x}, 0)$ for the specification $(\Omega(x,0), \mathcal{A}(x,0), \mathcal{P}(\cdot/x,0))$ can be determined through integrations similar to that in (4A). In this manner the conditional distributions are determined for all $\bar{x}$ points. Thus regularity is no problem.

Separability

It is important in the analysis of a stochastic process to consider many events that are specified over uncountable time sets.[8A] Events of this type are not normally in the σ-algebra $\mathcal{A}$ because it is impossible to express them in terms of countable set operations on sets known to be in $\mathcal{A}$ (vector intervals). If a stochastic process possesses the property of separability, it is usually possible to modify an event of this type on a set of probability zero in such a way that the event can be decomposed into countably many events in $\mathcal{A}$ and hence become an event in $\mathcal{A}$ itself. Separability may be defined as follows.

Definition: A stochastic process $\{X\}$ in R^N is separable if to every Borel set A in R^N there corresponds a sequence of time t_i and an ω set Λ_A of probability zero such that for $\omega \epsilon \Omega - \Lambda_A$

$$\left\{\omega / \bar{X}_\omega(t_i) \epsilon A,\ i \geq 1\right\} = \left\{\omega / \bar{X}_\omega(t) \epsilon A, t \epsilon T\right\}$$

8A. (a) - (f) in the subsection entitled, Remarks Concerning the Definition of a Stochastic Process, deal with events of this type.

If $\{\bar{X}\}$ is a regular process, it is possible to define separability with respect to the probability space $(\Omega(\bar{x}_1, t_1), \alpha(\bar{x}_1, t_1), \mathcal{P}(\cdot/\bar{x}_1, t_1))$ for the collection of sample functions that pass through the point $\bar{x}_1$ at t_1. This is done by substituting $\mathcal{P}(\cdot/x_1, t_1)$ in the above definition for $\mathcal{P}$.

The modification of events is illustrated by the following proposition.

Proposition: <u>If a regular stochastic process</u> $\{\bar{X}\}$ <u>in</u> R^N <u>is separable with respect to the probability space</u> $(\Omega(\bar{x},0), \alpha(\bar{x},0), \mathcal{P}(\cdot/\bar{x},0)$ <u>and</u> Γ <u>is a Borel set in</u> R^N, <u>there is a set</u> Λ_Γ <u>of probability zero such that the first passage time</u> $\tau(\bar{x})$ <u>defined for the process</u> $\{\bar{X}\}$ <u>by the equation</u>,

$$\tau_\omega(\bar{x}) = \begin{cases} \sup\left\{t \,\middle|\, \bar{X}_\omega(t^1) \epsilon \Gamma \text{ for } 0 < t^1 < t \text{ and } \bar{X}(0) = \bar{x}\right\}, \text{for } \omega \epsilon \Omega(x,0) - \Lambda_\Gamma \\ 0, \text{ for } \omega \epsilon \Lambda_\Gamma \end{cases}$$

is a random variable on the probability space $(\Omega(\bar{x},0), \alpha(\bar{x},0), \mathcal{P}(\cdot/\bar{x},0))$.

Proof: To show that $\tau(x)$ is a random variable all that must be shown is that $\{\omega/\tau_\omega(\bar{x}) > t\}$ is a set in $\alpha(\bar{x},0)$. This set may be expressed as

$$\{\omega/\tau_\omega(\bar{x}) > t\} = \{\omega/\bar{X}_\omega(t^1) \epsilon \Gamma, \text{ for } t^1 \epsilon (0,t]\} =$$

$$\bigcap_{t^1 \epsilon (0,t]} \{\omega/\bar{X}_\omega(t) \epsilon \Gamma\}$$

Separability makes it possible to equate the arbitrary intersection to a countable intersection on the complement of a set of measure zero. By letting Λ_Γ be this set of measure zero, $\tau(\bar{x})$ becomes a Baire function. This terminates the proof.

Under the definition given in this appendix for the probability space $(\Omega, \alpha, \mathcal{P})$, few processes of interest are separable. Such processes as the response of a structural system to white noise and Brownian motion are not separable. However, for any probability space of a stochastic process $(\Omega, \alpha, \mathcal{P})$ it is possible to derive a new probability space $(\Omega, \alpha, \mathcal{P})$ which will form a separable process with the random vectors $\{\bar{X}(t),\ t \in T\}$.[9A] α^1 and $\mathcal{P}^1$ are uniquely determined by α and $\mathcal{P}$ [29, p. 69]. Thus a stochastic process that is specified by the finite-dimensional distributions can always be considered separable.

9A. This is only true if R^N is constructed from R^1 by $\prod_{i=1}^{N} R^{1i}$ where $R^1 = [-\infty, \infty]$.

APPENDIX B

LANGEVIN'S EQUATION, THE FOKKER-PLANCK EQUATIONS, AND WHITE NOISE AS AN INPUT

In this appendix the use of the purely random process[1B,2B] (white noise) and the Langevin stochastic differential equation are discussed. The origin of the material presented here is the work of

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1B. The definition of a purely random process given by Wang and Uhlenbeck [17, pp. 324 and 332] is adopted in this work. If t_1 and t_2 are times between minus infinity and infinity, a process $\{Z\}$ is purely random if it has the following properties:

1. If $t_1 \neq t_2$ the random variables $Z(t_1)$ and $Z(t_2)$ are independent.
2. The expectations $E\,[Z(t_1)]$ and $E[Z(t_1)Z(t_2)]$ are

$$E\,[Z(t_1)] = 0$$

and

$$E\,[Z(t_1)Z(t_2)] = \delta(t_1 - t_2)\ a$$

where $\delta(t_1 - t_2)$ is the Dirac delta defined by the formal operation on a function f by

$$\int_{-\epsilon}^{\epsilon} f(x)\ \delta(x)dx = f(0)$$

for every $\epsilon > 0$ and a is a positive constant.

2B. Discussions of the physical justification for the use of the purely random process are given in [30, ch. 11] and [22 pp.39-40].

J. L. Doob [31, pp. 131-137], [5, pp. 351-369], and [4, pp. 269-272].

The problem that Doob considers is the statistical description of the response of a lumped parameter system (mechanical or electrical) that is excited by a random noise. Such systems can be described by a finite number of finite order ordinary differential equations. In this appendix the damped harmonic oscillator is taken as an example of such a system. The differential equation for it is

$$\frac{d^2x(t)}{dt^2} + \beta\frac{d\,x(t)}{dt} + \omega_0^2\,x(t) = f(t) \tag{1B}$$

$f(t)$ is the exciting force. $x(t)$ is the response (position). β and ω_0 are the damping factor and the natural frequency.

Looking at equation (1B) from a probabilistic viewpoint the following interpretation might appear pausible: To each sample function $F_{\omega}(t)$ of an input stochastic process $\{F\}$ (i.e., a collection of random variables $\{F(t),\ t\epsilon T\}$ and a probability space $(\Omega, \mathcal{a}, \mathcal{P}_{\{F\}})$ there corresponds a function $X_{\hat{\omega}}(t)$ which may be considered a sample function of a response process $\{X\}$ (i.e., a collection of random variables $\{X(t),\ t\epsilon T\}$ and a probability space $(\Omega, \mathcal{a}, \mathcal{P}_{\{X\}})$).[3B] If the construction of a stochastic process given in Appendix A is

3B. This interpretation is due to Langevin, and when dealing with stochastic processes equation (1B) is called the Langevin equation for the system [30, p. 438]. Since Ω is the same for both processes the hat is used on ω to indicate that the output sample function $X_{\hat{\omega}}(t)$ is not the same function (ω - point) as the input function.

considered, this interpretation is not quite correct. In this construction of a stochastic process Ω is the space of real-valued functions. Equation (1B) does not make sense for all such functions. The set of functions for which it makes no sense can be ignored if $\mathcal{P}_{\{F\}}$ is restricted so that this set has probability zero. (There is no logical reason for doing otherwise.) There is a similar problem in the specification of the output process $\{X\}$. Ω and α are the same as Ω and α for $\{F\}$. Obviously, all the real-valued functions do not have second derivatives. Thus, in any appropriate specification of $\mathcal{P}_{\{X\}}$ the set of sample functions which do not have second derivatives must be a set of probability zero.

Doob's Interpretation of the Langevin Equation

Physically, a sample path of a purely random process is a chain of impulsive forces. When a deterministic problem involving impulsive forces is considered, it is convenient to use a momentum equation rather than a force equation so that the change in acceleration across an impulse does not have to be considered. Doob employed this technique in his interpretation of the Langevin equation. He proposed the following interpretation [4, p. 272]: the concept of Langevin means that the integral equation

$$\int_{t_0}^{t_1} f(t)\,d\dot{X}_{\tilde{\omega}}(t) + \beta \int_{t_0}^{t_1} f(t)\,\dot{X}_{\tilde{\omega}}(t)\,dt + \omega_0^2 \int_{t_0}^{t_1} f(t)\,X_{\tilde{\omega}}(t)\,dt =$$

$$= \int_{t_0}^{t_1} f(t)\,dB_{\omega}(t) \tag{2B}$$

holds with probability one whenever $f(t)$ has a bounded derivative on the interval $[t_0, t_1]$. The integrals in equation (2B) will be given a precise mathematical meaning in the following discussion. Then after several results are obtained the manner in which Doob's approach relates to the physical problem being considered here is discussed.

The Process $\{B\}$

The process $\{B\}$ takes over the role of the input to the system; however, its relationship to the purely random process is not direct. The process $\{B\}$ is a regular, separable, temporally homogeneous differential process [29, ch. VIII]. (In the present work it will be referred to as a differential process.) It is defined by two properties:

(a) If $-\infty < t_1 < t_2 < \ldots < t_k < \infty$, then the random variables

$$B(t_2) - B(t_1), \ldots, B(t_k) - B(t_{k-1})$$

are mutually independent.[4B]

(b) The distribution function for the difference $B(s + \tau) - B(s)$ is independent of s.

Differential processes may be divided into two classes according

4B. The random variable $B(t)$ is not defined, and it is not needed in this discussion. Only differences of the form $B(t) - B(s)$ are considered. Doob [31, pp. 128-131] shows that a stochastic process may be defined in terms of the differences. Thus, it is possible to speak of a $\mathcal{P}_{\{B\}}$ measure for this process.

to the expectation

$$\sigma^2(\tau) = E\left[(B(s+\tau) - B(s))^2\right], \quad 0 < \tau < \infty$$

They are

(a) processes for which $\sigma^2(\tau) = \infty$

(b) processes for which $\sigma^2(\tau) < \infty$

If $\sigma^2(\tau) < \infty$, then $E[B(s+\tau) - B(s)]$ exists and is also less than infinity. This expectation is a measure of bias for the differential process. In this appendix it is taken to be zero. (There is no reason for biasing the input.) Then the function $\sigma^2(\tau)$ must satisfy the equation

$$\sigma^2(\tau + \eta) = \sigma^2(\tau) + \sigma^2(\eta) \tag{3B}$$

This is true because

$$\begin{aligned}
\sigma^2(\tau+\eta) &= E\left[(B(s+\tau+\eta) - B(s))^2\right] \\
&= E\left[(B(s+\tau+\eta) - B(s+\tau) + B(s+\tau) - B(s))^2\right] \\
&= E\left[(B(s+\tau+\eta) - B(s+\tau))^2\right] + E\left[(B(s+\tau) - B(s))^2\right] \\
&\quad + 2\,E\left[B(s+\tau+\eta) - B(s+\tau)\right]E\left[B(s+\tau) - B(s)\right] \\
&= \sigma^2(\eta) + \sigma^2(\tau)
\end{aligned}$$

The solution to equation (3B) is

$$\sigma^2(\tau) = \sigma^2 \tau$$

where σ^2 is a constant.

Thus the two alternatives are

$$\text{(a)} \quad \sigma^2(\tau) = \infty$$

$$\text{(b)} \quad \sigma^2(\tau) = \sigma^2 \tau$$

In either case, if $f(t)$ has a bounded derivative on the closed interval $[t_0, t_1]$, $-\infty < t_0 < t_1 < \infty$, the integral

$$\int_{t_0}^{t_1} f(t) \ dB_\omega(t)$$

exists in the R-S sence[5B] for all sample functions except an exceptional set of probability zero [28, pp. 131-138]. This is true even though the set of sample functions of a differential process that are not of bounded-variation is a set of probability one.

5B. R-S sense means that the usual Riemann-Stieltjes sums for the integral converge.

The Solution to Doob's Integral Equation

Using only integration by parts and algebraic manipulation, it is possible to obtain a solution to equation (19B). It is

$$\dot{X}_{\tilde{\omega}}(t) = \frac{1}{\lambda_1-\lambda_2}\left\{\begin{array}{l} (\lambda_1 e^{-\lambda_1 t} - \lambda_2 e^{-\lambda_2 t})\ \dot{X}_{\tilde{\omega}}(0) \\ \\ + \lambda_1\lambda_2\ (e^{-\lambda_2 t} - e^{-\lambda_1 t})\ X_{\tilde{\omega}}(0) \\ \\ \int_0^t (\lambda_1 e^{-\lambda_1(t-\tau)} - \lambda_2 e^{-\lambda_2(t-\tau)})\, dB_{\omega}(\tau) \end{array}\right. \tag{4B}$$

and

$$X_{\hat{\omega}}(t) = \frac{1}{\lambda_1-\lambda_2}\left\{\begin{array}{l} (e^{-\lambda_2 t} - e^{-\lambda_1 t})\ \dot{X}_{\hat{\omega}}(0) \\ \\ + (\lambda_1 e^{-\lambda_2 t} - \lambda_2 e^{-\lambda_1 t})\ X_{\hat{\omega}}(0) \\ \\ + \int_0^t (e^{-\lambda_2(t-\tau)} - e^{-\lambda(t-\tau)})\, dB_{\omega}(\tau) \end{array}\right. \tag{5B}$$

where $\lambda_1 = \frac{\beta}{2} + i\omega_1, \lambda_2 = \frac{\beta}{2} - i\omega_1$, and $\omega_1^{\ 2} = \omega_0^{\ 2} - \frac{\beta^2}{4} = (i = \sqrt{-1})$[6B]

Implications of the Solution

Equations (4B) and (5B) illustrate the fact that the response

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6B. Only the under-damped oscillator is considered. The over-damped and the critically damped may be handled in a similar manner.

should be a vector Markov process in a position-velocity space. The set of sample functions of the process $\{B\}$ for which these equations hold is a set of probability one. Therefore, the set of sample functions of the form $(X_{\tilde{\omega}}(t), \dot{X}_{\tilde{\omega}}(t))$ in the response process for which equations (4B) and (5B) hold must be a set of probability one. Thus the random vector $(X(t), \dot{X}(t))$ only depends on the random vector $(X(0), \dot{X}(0))$ and the integral $\int_0^t e^{-\lambda_{1,2}(t-\tau)} dB(t)$. Writing the integral as the limit of Riemann-Stieltjes sums, the differences $B(t_i) - B(t_j)$ used in these sums are independent of any time before $t = 0$. Thus, the random vector $(X(t), \dot{X}(t))$ should be dependent only on $(X(0), \dot{X}(0))$ and not on $(X(t^1), \dot{X}(t^1))$ for $t^1 < 0$. The choice of $t = 0$ was arbitrary. Equations (4B) and (5B) could be rewritten in terms of any initial time t_0, and this argument would still apply.

The integrals in the equation (4B) and (5B) can be written as

$$\int_0^t (\lambda_1 e^{-\lambda_1(t-\tau)} - \lambda_2 e^{-\lambda_2(t-\tau)}) dB_{\omega}(\tau)$$

$$= (\lambda_1 - \lambda_2)\left[B_{\omega}(t) - B_{\omega}(0)\right] - \int_0^t (\lambda_1^2 e^{-\lambda_1(t-\tau)} + \lambda_2^2 e^{-\lambda_2(t-\tau)}) \tag{6B}$$

$$\times \quad B(\tau) - B(0) d\tau$$

and

$$\left.\begin{aligned} &\int_0^t \left(e^{-\lambda_1(t-\tau)} - e^{-\lambda_2(t-\tau)}\right) dB_{\omega}(\tau) \\ &= \int_0^t \left(\lambda_1 e^{-\lambda_1(t-\tau)} - \lambda_2 e^{-\lambda_2(t-\tau)}\right) \left[B_{\omega}(\tau) - B_{\omega}(0)\right] d\tau \end{aligned}\right\} \tag{7B}$$

respectively. The integrals imply for the sample functions $(X_{\tilde{\omega}}(t),$ $\dot{X}_{\tilde{\omega}}(t))$ of the response corresponding to the samples $B_{\omega}(t)$ of the differential process that

(a) $X_{\tilde{\omega}}(t)$ and $\dot{X}_{\tilde{\omega}}(t)$ are continuous if $B_{\omega}(t)$ is continuous.

(b) $\dot{X}_{\tilde{\omega}}(t)$ is differentiable if $B_{\omega}(t)$ is differentiable.

The Fokker-Planck Equations

Let $p(y,\dot{y},s/x,\dot{x},r)$ be the conditional probability density function for the random vector $(X(s),\dot{X}(s))$, given that the random vector $(X(r),\dot{X}(r)) = (x,\dot{x})$, for all times r and s where $r < s$. The Markov property, which is discussed in the previous section, implies that this density satisfies a Chapman-Kolmogorov equation. It is

$$P(z,\dot{z},t/x,\dot{x},r) = \int_{-\infty}^{\infty}\int p(z,\dot{z},t/y,\dot{y},s)\; p(y,\dot{y},s/x,\dot{x},r)\; dy d\dot{y} \tag{8B}$$

where $r < s < t$.

A special form of conditional moment may be considered. It is

$$A_{m,n}(x,\dot{x},r) = \lim_{s \to r^+} \frac{1}{s-r} \int_{-\infty}^{\infty}\!\!\int (\dot{y}-\dot{x})^m (y-x)^n p(y,\dot{y},s/x,\dot{x},r)\,dy\,d\dot{y}$$

$$= \lim_{s \to r^+} \frac{1}{s-r}\; E\left[(\dot{X}(s)-\dot{X}(r))^m (X(s)-X(r))^n \,\middle|\, \begin{matrix} X(r) = x \\ \dot{X}(r) = \dot{x} \end{matrix} \right]$$

Equations (4B) and (5B) can be used to calculate these moments. If $\sigma^2(t)$ were infinite, then equation (4B) and (6B) indicate that some of the $A_{m,n}(x,\dot{x},r)$ would also be infinite. Thus, it is assumed that $\sigma^2(t) < \infty$. Then the calculations for $A_{0,1}(x,\dot{x},r)$, $A_{1,0}(x,\dot{x},r)$, $A_{1,1}(x,\dot{x},r)$, $A_{0,2}(x,\dot{x},r)$ and $A_{2,0}(x,\dot{x},r)$ are[6B]

$$A_{0,1}(x,\dot{x},r) = \lim_{s \to 0^+} \frac{1}{s}\; E\left[X(s) - X(r) \,\middle|\, \begin{matrix} X(0) = x \\ \dot{X}(0) = \dot{x} \end{matrix} \right]$$

$$= \lim_{s \to 0^+} \left\{ \begin{aligned} &\frac{e^{-\lambda_2 s} - e^{-\lambda_1 s}}{(\lambda_1-\lambda_2)s}\,\dot{x} + \frac{(\lambda_1 e^{-\lambda_2 s} - \lambda_2 e^{-\lambda_1 s}) - (\lambda_1-\lambda_2)}{(\lambda_1-\lambda_2)s}\,x \\ &+ \frac{1}{(\lambda_1-\lambda_2)s} \int_0^s (\lambda_1 e^{-\lambda_1(s-\tau)} - \lambda_2 e^{-\lambda_1(s-\tau)})\,E\big[B(\tau)-B(0)\big]\,d\tau \end{aligned} \right.$$

6B. The parameter r is set equal to zero in these calculations; so that the form of (4B) and (5B) is not unnecessarily complicated. For $r \neq 0$ the calculation can be reduced to the expressions that appear here and identical results are obtained.

$$= \lim_{s \to 0^+} \left[\frac{e^{-\lambda_2 s} - e^{-\lambda_1 s}}{(\lambda_1 - \lambda_2) s} \dot{x} + \left(\frac{\lambda_1 (e^{-\lambda_2 s} - 1)}{(\lambda_1 - \lambda_2) s} - \frac{\lambda_2 (e^{\lambda_2 s} - 1)}{(\lambda_1 - \lambda_2) s} \right) x \right] = \dot{x}$$

$$A_{1,0} \; (x, \dot{x}, r) = \lim_{s \to 0^+} \frac{1}{s} E\left[\dot{X}(s) - \dot{X}(0) \middle| \begin{matrix} X(0) = x \\ \dot{X}(0) = \dot{x} \end{matrix} \right]$$

$$= \lim_{s \to 0^+} \begin{cases} \dfrac{\lambda_1 (e^{-\lambda_1 s} - 1) - \lambda_2 (e^{-\lambda_2 s} - 1)}{(\lambda_1 - \lambda_2) s} \dot{x} + \\[2ex] + \dfrac{\lambda_2 \lambda_1 \; (e^{-\lambda_2 s} - e^{-\lambda_1 s})}{(\lambda_1 - \lambda_2) s} x + \\[2ex] + \dfrac{1}{(\lambda_1 - \lambda_2) s} \displaystyle\int_0^s (\lambda_1{}^2 e^{-\lambda_1 (s-\tau)} - \lambda_2{}^2 e^{-\lambda_2 (s-\tau)}) \; E\left[B(t) - B(0) \right] d\tau + \\[2ex] + \dfrac{\lambda_1 - \lambda_2}{(\lambda_1 - \lambda_2) s} \; E\left[B(t) - B(0) \right] \end{cases}$$

$$= - \; (\lambda_1 + \lambda_2) \; \dot{x} - \lambda_1 \lambda_2 x$$

The quantities $\lambda_1 + \lambda_2$ and $\lambda_1 \lambda_2$ are

$$\lambda_1 + \lambda_2 = \frac{\beta}{2} + i\,\omega_1 + \frac{\beta}{2} - i\,\omega_1 = \beta$$

and

$$\lambda_1 \lambda_2 = (\frac{\beta}{2} + i\,\omega_1) \quad (\frac{\beta}{2} - i\,\omega_1) = \frac{\beta^2}{4} + \omega_1{}^2 = \omega_0{}^2$$

Thus

$$A_{1,0}(x,\dot{x},r) = -\beta\dot{x} - \omega_0^2 x$$

$$A_{11}(x,\dot{x},r) = \lim_{s\to 0^+} \frac{1}{S} E\left[(\dot{X}(s) - \dot{X}(0))(X(s) - X(0)) \middle| \begin{matrix} X(0) = x \\ \dot{X}(0) = \dot{x} \end{matrix}\right] =$$

$$= \lim_{s\to 0^+} E\left[\left[\frac{\lambda_1(e^{-\lambda_1 s}-1) - \lambda_2(e^{-\lambda_2 s}-1)}{\lambda_1-\lambda_2\ s}\dot{x} + \frac{\lambda_1\lambda_2(e^{-\lambda_1 s} - e^{-\lambda_2 s})}{(\lambda_1-\lambda_2)\ s} x + \frac{1}{(\lambda_1-\lambda_2)\ s}\int_0^s (\lambda_1^2 e^{-\lambda_1(s-\tau)} - \lambda_2^2 e^{-\lambda_2(s-\tau)})(B(\tau) - B(0))\, d\tau\right] \times \left[\frac{e^{-\lambda_2 s} - e^{-\lambda_1 s}}{(\lambda_1-\lambda_2)}\dot{x} + \frac{(\lambda_1 e^{-\lambda_2 s} - \lambda_2 e^{-\lambda_1 s}) - (\lambda_1-\lambda_2)}{(\lambda_1-\lambda_2)} x + \frac{1}{(\lambda_1-\lambda_2)}\int_0^s (\lambda_1 e^{-\lambda_1(s-\tau)} - \lambda_2 e^{-\lambda_2(s-\tau)})(B(\tau) - B(0))\, d\tau\right]\right]$$

$$= 0$$

$$A_{0,2}(x,\dot{x},r) = \lim_{s \to 0^+} \frac{1}{s} E\left[(X(s) - X(0))^2 \,\middle|\, \begin{matrix} X(0) = x \\ \dot{X}(0) = \dot{x} \end{matrix} \right] =$$

$$= \lim_{s \to 0^+} \frac{1}{s} E\left[\left(\frac{e^{-\lambda_2 s} - e^{\lambda_1 s}}{(\lambda_1-\lambda_2)} \dot{x} + \frac{(\lambda_1 e^{-\lambda_2 s} - \lambda_2 e^{-\lambda_1 s}) - (\lambda_1-\lambda_2)}{(\lambda_1-\lambda_2)} x + \frac{1}{(\lambda_1-\lambda_2)} \int_0^s (\lambda_1 e^{-\lambda_1(s-\tau)} - \lambda_2 e^{-\lambda_2(s-\tau)}) \quad B(\tau) - B(0) \, d\tau \right)^2 \right] =$$

$$= 0$$

$$A_{2,0}(x,\dot{x},r) = \lim_{s \to 0^+} \frac{1}{s} E\left[(\dot{X}(s) - \dot{X}(0))^2 \,\middle|\, \begin{matrix} X(r) = x \\ \dot{X}(r) = \dot{x} \end{matrix} \right] =$$

$$= \lim_{s \to 0^+} \frac{1}{s} E\left[\left\{ \frac{\lambda_1(e^{-\lambda_1 s}-1) - \lambda_2(e^{-\lambda_2 s}-1)}{(\lambda_1-\lambda_2)} \dot{x} + \frac{\lambda_2\lambda_1(e^{-\lambda_2 s} - e^{-\lambda_1 s})}{(\lambda_1-\lambda_2)} x + \frac{1}{\lambda_1-\lambda_2} \int_0^s (\lambda_1^2 e^{-\lambda_1(s-\tau)} - \lambda_2^2 e^{-\lambda_2(s-\tau)})(B(\tau)-B(0))d\tau + B(s) - B(0) \right\}^2 \right] =$$

$$= \lim_{s \to 0^+} \frac{E\left[(B(s) - B(0))^2\right]}{s} = \lim_{s \to 0^+} \sigma^2 \frac{s}{s} = \sigma^2$$

In order to calculate the $A_{m,n}(x,\dot{x},r)$'s for $m + n \geq 3$ an additional assumption is required. It is that the response must be continuous in velocity with probability one (i.e., the set of sample paths discontinuous in velocity or displacement must have probability zero). Thus, if the differential process is to be a meaningful input, its set of discontinuous sample paths must have probability zero. Lévy has shown [32, pp. 166-167] the the only separable differential process for which the set of continuous sample paths has $\mathcal{P}_{\{B\}}$ -measure one is the differential process for which the difference $B(t + s) - B(s)$ has a Gaussian distribution and $\sigma^2(t) < \infty$.[7B] In this case $E\left[(B(s)-B(0))^n\right]$ is

$$E\left[(B(s) - B(0))^n\right] = \begin{cases} 0, & n = 1,3,5,\ldots \\ \dfrac{n!}{2^{n/2}\left(\frac{n}{2}\right)!}\,\sigma^n\, t^{n/2}, & n = 2,4,6,\ldots \end{cases}$$

7B. In many discussions of Fokker-Planck equations the Lindéberg condition, which is

$$\lim_{s \to r^+} \iint_{|(y-x,\dot{y}-\dot{x})| \geq \delta} p(y,\dot{y},s/x,\dot{x},r)\; dy d\dot{y} = 0$$

where $|(a,b)| = \sqrt{a^2 + b^2}$ and $\delta > 0$ is used to show that the $A_{m,n}(x,\dot{x},r)$ vanisn for $m + n \geq 3$. D. Ray has shown that the Lindéberg condition and a condition on the smoothness of $p(y,\dot{y},s/x,\dot{x},r)$ imply continuity of the sample paths with probability one [21, p. 475]. Thus, the Lindéberg condition implies that the process $\{B\}$ must be a Gaussian differential process with $\sigma(t) < \infty$. Equations (4B) and (5B) may be used to show that if $\{B\}$ is a Gaussian differential process then the Lindéberg condition is satisfied. Therefore, the Lindéberg condition and the assumption of sample path continuity that is used here are equivalent.

This formula and 4B and 5B may be used to show that $A_{m,n}(x,\dot{x},r) = 0, m + n \geq 3$.

If $p(y,\dot{y},s/x,\dot{x},r)$ is assumed to be twice differentiable in $\dot{y},\dot{y}$, x, and x and since $A_{0,1}(x,\dot{x},r)$, $A_{1,0}(x,\dot{x},r)$, and $A_{1,1}(x,\dot{x},r)$ are differentiable functions and $A_{0,2}(x,\dot{x},r)$ and $A_{2,0}(x,\dot{x},r)$ are twice differentiable functions in x and $\dot{x}$, the Chapman-Kolmogorov equation, equation (8B), may be used to derive forward and backward Fokker-Planck equations which $p(y,\dot{y},s/x,\dot{x},r)$ must satisfy [7, X.4, X.5, and X.6] and [28, sec. 10.3-2]. The backward equation is

$$\left\{ \frac{\partial}{\partial r} + L_2^{*} \right\} p(y,\dot{y},s/x,\dot{x},r) = 0 \tag{9B}$$

The forward equation is

$$\left\{ \frac{\partial}{\partial s} - L_2 \right\} p(y,\dot{y},s/x,\dot{x},r) = 0$$

Here L_2^{*} is the operator

$$L_2^{*} = \left\{ \begin{array}{l} A_{1,0}(x,\dot{x},r) \dfrac{\partial}{\partial \dot{x}} + A_{0,1}(x,\dot{x},r) \dfrac{\partial}{\partial x} + A_{1,1}(x,\dot{x},r) \dfrac{\partial^2}{\partial x \partial \dot{x}} + \\[2ex] + \dfrac{1}{2} A_{0,2}(x,\dot{x},r) \dfrac{\partial^2}{\partial x^2} + \dfrac{1}{2} A_{2,0}(x,\dot{x},r) \dfrac{\partial^2}{\partial \dot{x}^2} \end{array} \right.$$

and L_2 is the operator

$$L_2 = \begin{cases} -\frac{\partial}{\partial \dot{y}} (A_{1,0}(y,\dot{y},s)\ \cdot) - \frac{\partial}{\partial y} (A_{0,1}(y,\dot{y},s)\ \cdot) + \frac{\partial^2}{\partial y \partial \dot{y}} (A_{1,1}(y,\dot{y},s)\ \cdot) + \\ + \frac{1}{2} \frac{\partial^2}{\partial \dot{y}^2} (A_{2,0}(y,\dot{y},s)\ \cdot) + \frac{1}{2} \frac{\partial^2}{\partial y^2} (A_{0,2}(y,\dot{y},s)\ \cdot) \end{cases}$$

A Solution to the Fokker-Planck Equations

A solution for $p(y,\dot{y},s/x,\dot{x},r)$ may be obtained from equations (9B) and (10B). To seek this solution an initial condition is imposed on $p(y,\dot{y},s/x,\dot{x},r)$. It is

$$\lim_{s \to r^+} p(y,\dot{y},s/x,\dot{x},r) = \delta(y-x)\ \delta(\dot{y}-\dot{x}) \tag{11B}$$

where $\delta(y-x)$ is the Dirac delta function; so that, (11B) means

$$\lim_{s \to r^+} \iint_{|(y-x,\dot{y}-\dot{x})| < \epsilon} p(y,\dot{y},s/x,\dot{x},r)\ dy d\dot{y} = 1$$

for any $\epsilon > 0$. The connection between this condition and the Lindéberg condition is obvious. From Ray's work (footnote 7B), equation (11B) could be considered to be a result of sample path continuity. Also, a boundary condition is imposed. The boundary condition on equation (9B) is

$$\lim_{|(x,\dot{x})| \to \infty} p(y,\dot{y},s/x,\dot{x},r) = 0 \tag{12B}$$

The boundary condition for equation (10B) is

$$\lim_{|(y,\dot{y})| \to \infty} p(y,\dot{y},s/x,\dot{x},r) = 0 \tag{13B}$$

Equation (13B) expresses the fact that the probability of starting from a finite value $(x,\dot{x})$ and reaching infinity in a finite amount of time is remote. On the other hand, equation (14B) expresses the fact that the probability of starting infinitely far away from $(x,\dot{x}) = (0,0)$ and reaching a finite value in a finite amount of time is remote.

An expression for $p(y,\dot{y},s/x,\dot{x},r)$ which satisfies the forward and backward equations is given by Wang and Uhlenbeck [17, p. 335]. It is

$$p(y,\dot{y},s/x,\dot{x},r) =$$

$$\frac{\exp\left[-\frac{1}{2(1-\rho_t^2)}\left(\frac{(y-C_t)^2}{\sigma_t^2} - \frac{2\rho_t(y-C_t)(\dot{y}-\dot{C}_t)}{\sigma_t\dot{\sigma}_t} + \frac{(\dot{y}-\dot{C}_t)^2}{\dot{\sigma}_t^2}\right)\right]}{2\pi\,\sigma_t\,\dot{\sigma}_t\,(1-\rho_t^2)^{1/2}} \tag{14B}$$

where

$$t = s-r$$

$$C_t = \dot{x}\,\frac{\exp\left[-\frac{1}{2}\beta t\right]\sin\omega_1 t}{\omega_1} + x\,\frac{\exp\left[-\frac{1}{2}\beta t\right]\left({}_1\cos\omega_1 t + \beta/2\,\sin\omega_1 t\right)}{\omega_1}$$

$$\dot{C}_t = \dot{x}\,\frac{\exp\left[-\frac{1}{2}\beta t\right](\omega_1\cos\omega_1 t - \beta/2\,\sin\omega_1 t)}{\omega_1} + \frac{\omega_0^2\exp\left[-\frac{1}{2}\beta t\right]\sin\omega_1 t}{\omega_1}$$

$$\sigma_t^{\ 2} = \frac{\sigma^2}{2\beta\omega_0^{\ 2}}\left[1 - \frac{1}{\omega_1^{\ 2}}\exp\left[-\beta t\right](\omega_1^{\ 2} + \tfrac{1}{2}\beta^2\sin^2\omega_1 t + \beta\omega_1\sin\omega_1 t\cos\omega_1 t)\right]$$

$$\dot{\sigma}_t^{\ 2} = \frac{\sigma^2}{2\beta}\left[1 - \frac{1}{\omega_1^{\ 2}}\exp\ -\beta t\right](\omega_1^{\ 2} + \tfrac{1}{2}\beta^2\sin^2\omega_1 t - \beta\omega_1\sin\omega_1 t\cos\omega_1 t)\Big]$$

$$\rho_t = \frac{\sigma^2\exp\left[-\beta t\right]\sin^2\omega_1 t}{2\omega_1^{\ 2}\,\sigma_t\,\dot{\sigma}_t}$$

If the limit $s - r \to \infty$ is taken, the function $p(y,\dot{y},s/x,\dot{x},r)$ becomes a function of y and $\dot{y}$ only. This function is

$$p(y,\dot{y}) = \frac{\exp\left[-\frac{1}{2}\left(\frac{y^2}{\sigma_\infty^{\ 2}} + \frac{\dot{y}^2}{\dot{\sigma}_\infty^{\ 2}}\right)\right]}{2\pi\,\sigma_\infty\,\dot{\sigma}_\infty} \tag{15B}$$

where $\sigma_\infty^{\ 2} = \dfrac{\sigma^2}{2\beta\omega_0^{\ 2}}$ and $\dot{\sigma}_\infty^{\ 2} = \dfrac{\sigma^2}{2\beta}$.

If the oscillator is considered to have been set into motion at time $t = -\infty$, this function can be viewed as the density function for the random vector $(X(t_0),\ \dot{X}(t_0))$ at an arbitrary time t_0 in the interval $(-\infty, \infty)$. To be consistent with the notation of the random vector $(X(t_0),\ \dot{X}(t_0))$ this function will be written $p(x_0,\dot{x}_0,t_0)$ even though it is not a function of time.

Since t_0 is arbitrary, it should be noted that for $p(x_0,\dot{x}_0,t_0)$ to be consistent with the process $\left\{(X,\dot{X})\right\}$ the following equation must be satisfied:

$$p(x_1,\dot{x}_1,t_1) = \iint_{-\infty}^{\infty} p(x_1,\dot{x}_1,t/x_0,\dot{x}_0,t_0)\ p(x_0,\dot{x}_0,t)\ dx_0 d\dot{x}_0 \tag{16B}$$

That (15B) satisfies this equation is easily verified.

From (14B) it can be seen that $p(y,\dot{y},s/x,\dot{x},r)$ is a function of the time difference $s - r$, and from (15B) it can be seen that $p(x,\dot{x},t)$ is independent of time. Thus, from the method given for the determination of the measure $\mathcal{P}_{\{\bar{y}\}}$ in Appendix A, each distribution function will be invariant under a time shift (i.e., $F(x_1,\dot{x}_1,t;\ldots;x_k,\dot{x}_k,t) = F(x_1,\dot{x}_1,t+\tau;\ldots;x_k,\dot{x}_k,t+\tau)$). This means that the response of such a system is a stationary process.

The Relationship Between the Differential Process and the Purely Random Process

To understand the relation between the differential process and the purely random process the physical interpretation of each must be considered. Consider equation (2B) and let $f(t) = 1$; integration by parts yields

$$\dot{X}_{\tilde{\omega}}(t) - \dot{X}_{\tilde{\omega}}(0) = -\beta\int_0^t \dot{X}_{\tilde{\omega}}(\tau)d\tau - \omega_0^{\,2}\int_0^t X_{\tilde{\omega}}(\tau)d\tau + B_{\omega}(t) - B_{\omega}(0)$$

Then $B_{\omega}(t) - B_{\omega}(0)$ represents the impulse due to some force. If $B_{\omega}(t)$ were differentiable (The fact that it is not has been discussed.), its derivative could be thought of as a sample path of a process which represents the force and

$$B_\omega(t) - B_\omega(0) = \int_0^t F_\omega(\tau)d\tau$$

For the variables $B(t_1) - B(t_0)$ and $B(t_2) - B(t_1)$ to be independent for $t_0 < t_1 < t_2$, $F(t)$ and $F(s)$ would have to be independent for for $t \neq s$. In addition to this it should be noted that

$$\sigma^2 s = E\left[(B(t + s) - B(t))^2\right]$$

$$= E\left[\int_s^{t+s} F(\tau)\ F(\tau^1)\ d\tau d\tau^1\right]$$

If the interchange of time integrations and expectation were valid, this would imply

$$\sigma^2 s = \int_t^{t+s} \int_t^{t+s} E\left[F(\tau)\ F(\tau^1)\right] d\tau d\tau^1$$

Also this would mean that, for every $s > 0$ and t in $(-\infty, \infty)$,

$$0 = E\left[B(t + s) - B(t)\right] = \int_t^{t+s} E\left[F(\tau)\right] = 0$$

Thus it is reasonable to assume that $E\left[F(\tau)\right] = 0$. These facts indicate that the representation

$$E\left[F(t)\ F(\tau)\right] = \delta\ (t-\tau)\ \sigma^2$$

would be appropriate.

The relationship between F(t) described here and the definition of a purely random process (footnote 1B) is obvious. The constant σ^2 equals a. Then the purely random process describes the derivative of the Gaussian differential process. Of course, the set of sample functions of the Gaussian differential process that have derivatives is a set of probability zero.

APPENDIX C

THE VELOCITY OF FIRST PASSAGES

The purpose of this appendix is to prove the following Proposition about first passage time velocities:

Proposition: Let $\{(X,\dot{X})\}$ be a regular, separable stochastic process on a position-velocity space, and let Γ_1 be the region

$$\Gamma_1 = \{(x,\dot{x})/-\infty < x \leq b,\ -\infty < \dot{x} < \infty\}$$

Suppose that the set of continuous sample functions is a set of probability one. Then the function $\dot{X}(\tau(x,\dot{x}))$, given in the definition that follows, for the velocities of first passages from the region Γ_1 is a random variable on the probability space $(\Omega(x,\dot{x}),\ \mathcal{a}(x,\dot{x}),\ \mathcal{P}(\cdot/x,\dot{x}))$.

Definition:[1C] $\dot{X}(\tau(x,\dot{x}))$ is defined by considering the following alternatives for the sample functions $(X_\omega(t),\ \dot{X}_\omega(t))$ that have $(X_\omega(0),\dot{X}_\omega(0)) = (x,\dot{x})$:

(a) If $(X_\omega(t),\ \dot{X}_\omega(t))$ is not continuous on every finite interval $[0,t^1]$,

1C. Since the set of continuous sample functions is a set of probability ones, $\dot{X}(\tau(x,\dot{x}))$ need not be defined meaningfully outside this set.

<u>then</u> $\dot{X}(\tau(x,\dot{x}))$ <u>is set equal to zero</u>.

(b) <u>If</u> $(X_\omega(t), \dot{X}_\omega(t))$ <u>is continuous on every finite interval</u> $[0,t^1]$ <u>but the level</u> b <u>is never reached</u>, $\dot{X}_\omega(\tau_\omega(x,\dot{x}))$ <u>is set equal to zero</u>.

(c) <u>If</u> $(X_\omega(t), \dot{X}_\omega(t))$ <u>is continuous on every finite interval</u> $[0,t^1]$ <u>and</u> $\tau_\omega(x,\dot{x}) = t^* < \infty$, <u>then</u> $\dot{X}_\omega(\tau_\omega(x,\dot{x}))$ <u>is set equal to</u> $\dot{X}(t^*)$.

<u>Proof of the Proposition About $\dot{X}(\tau(x,\dot{x}))$</u>

What must be shown is that the set $\{\omega / \dot{X}_\omega(\tau\ (x,\dot{x})) \leq \dot{z}\}$ is a member of the σ-algebra $\mathcal{a}(x,\dot{x})$. There are three cases to be considered:

(i) $(\dot{z} < 0)$: If $(X_\omega(t), \dot{X}_\omega(t))$ is continuous on every finite interval $[0,t]$, it is impossible to have a first passage of the level b with a negative velocity. Therefore, the set $\{\omega / \dot{X}_\omega(\tau_\omega(x,\dot{x})) \leq \dot{z}\}$ is the empty set, and hence, it is in $\mathcal{a}(x,\dot{x})$.

(ii) $(\dot{z} = 0)$: Either $(X_\omega(t), \dot{X}_\omega(t))$ is discontinuous in the interval $[0,\infty]$, $\dot{X}_\omega(\tau_\omega(x,\dot{x})) = 0$, or a first passage never occurs and $\tau_\omega(x,\dot{x}) = \infty$.

(a) The set of sample functions that are continuous on the interval $[0,\infty)$ is in $\mathcal{a}(x,\dot{x})$. Therefore, the set of sample functions that are not continuous is in $\mathcal{a}(x,\dot{x})$.

(b) The set of ω points for which $\tau_\omega(x,\dot{x}) = \infty$ is in $\mathcal{a}(x,\dot{x})$ because $\tau(x,\dot{x})$ is a random variable.

(c) Let A be the union of the sets discussed in (a) and

(b) above. The complement of A, denoted A^c, is the set of functions that start at $(x,\dot{x})$, are continuous over the interval $[0,\infty)$, and have first passage times less than infinity. The set of continuous sample functions which have first passages with zero velocity is a subset of A^c. It may be written as

$$A^c \left\{\omega \mid \dot{X}_\omega(\tau_\omega(x,\dot{x})) = 0\right\} =$$

$$A^c \cap \left[\bigcup_{m\geq 1} \left[\bigcap_{n\geq m} \left[\bigcup_{j=1}^{n^2} \begin{bmatrix} \bigcap_{k=1}^{j-1} \left\{\omega \middle| X_\omega\left(\frac{k}{n}\right) \le b\right\} \\ \cap\left(\bigcup_{r\in R_{j,n}} \left\{\omega \middle| b < X_\omega(r)\right\}\right) \\ \cap\left(\bigcup_{r^1 \in R_{j,n}} \left\{\omega \middle| \frac{1}{n} < \dot{X}_\omega(r^1) \le \frac{1}{n}\right\}\right) \end{bmatrix} \right]\right]\right]$$

where $R_{j,n}$ is the set of rationals in $\left[\frac{j-1}{n}, \frac{j}{n}\right]$. (The idea here is to use continuity to express the behavior of the sample functions at arbitrary values of time in terms of their behavior at rational values of time.)

To see that the set relation is true a sample function which has a first passage at time t^1 may be considered. For each n there is a j such that $t^1 \in \left(\frac{j-1}{n}, \frac{j}{n}\right]$.

Suppose $\dot{X}(t^1)$ is equal to zero. By the definition of a first passage there is a $t \in (t^1, \frac{j}{n}]$ such that $X(t) > b$. Thus continuity implies the existence of a rational $r \in (t^1, \frac{j}{n}]$ such that $X(r) > b$. Because $\dot{X}(t^1) = 0$, continuity implies the existence of a rational r^1 in $\left[\frac{j-1}{n}, \frac{j}{n}\right]$ such that $-\frac{1}{n} < \dot{X}(r^1) \leq \frac{1}{n}$.

Suppose $\dot{X}(t^1)$ is not zero. Then there is an n such that $n < \dot{X}(t)$ for every $t \in \left[\frac{j-1}{n}, \frac{j}{n}\right]$.

Each of the sets in (a), (b), and (c) above is in $\mathcal{a}\,(x,\dot{x})$. Thus, their union is in $\mathcal{a}\,(x,\dot{x})$.

(iii) $(\dot{z} > 0)$: Since $\left\{\omega \middle| \dot{X}_\omega(\tau_\omega(x,\dot{x})) \leq 0\right\}$ is in $\mathcal{a}\,(x,\dot{x})$ from (ii) all that remains to be shown is that $\left\{\omega \middle| 0 < \dot{X}_\omega(\tau_\omega(x,\dot{x})) \leq \dot{z}\right\}$ is in $\mathcal{a}\,(x,\dot{x})$. The argument used in (ii), part (c), may be used to show that $A^c \left\{\omega \middle| 0 \leq \dot{X}_\omega(\tau_\omega(x,\dot{x})) \leq \dot{z}\right\}$ is in $\mathcal{a}\,(x,\dot{x})$ simply by replacing

$$\left\{\omega \middle| -\frac{1}{n} < \dot{X}(r) \leq \frac{1}{n}\right\} \text{ by } \left\{\omega \middle| -\frac{1}{n} < \dot{X}(r) \leq \dot{z}+\frac{1}{n}\right\}.$$

Then $\left\{\omega \middle| 0 < \dot{X}_\omega(\tau_\omega(x,\dot{x})) \leq \dot{z}\right\}$ is the intersection of $A^c \left\{\omega \; 0 \leq \dot{X}_\omega(\tau_\omega(x,\dot{x})) \leq \dot{z}\right\}$ with the complement of $\left\{\omega / \dot{X}_\omega(\tau_\omega(x,\dot{x})) \leq 0\right\}$. Hence it is in $\mathcal{a}\,(x,\dot{x})$. This concludes the proof.

APPENDIX D

A DISCUSSION OF J. R. RICE'S ANALYSIS OF THE FIRST PASSAGE PROBLEM

The purpose of this appendix is to derive the basic equation of J. R. Rice's analysis of the first passage problem [15, p. 25, eq. II12] from the integral equation

$$N^{+}(t/x,\dot{x},0) = f_{\Gamma_1}(x,\dot{x},t) + \int_0^t \int_0^\infty N^{+}(t/b,\dot{y},\eta)\; g_{\Gamma_1}(\dot{y}/\eta)\; f_{\Gamma_1}(x,\dot{x},\eta)\,d\dot{y}\,d\eta \quad (1D)$$

and the assumption that the conditional density $g_{\Gamma_1}(\dot{y}/\eta)$ takes the form $g^{B}_{\Gamma_1}(\dot{y}/\eta)$,

$$g^{B}_{\Gamma_1}(\dot{y}/\eta) = \frac{\dot{y}\; p(b,\dot{y})}{N^{+}}$$

Accounts of J. R. Rice's work appear in [15] and [33]. Therefore, only a short summary of his research is given here. The problem which Rice considered may be stated as follows: consider the sample paths that start from an initial randomly located point $(X,\dot{X})$ that is distributed in Γ_1 (the half plane below $x = b$) according to the first order density function $p(x,\dot{x})$. Find the probability density function for the random variable for the first passage from Γ_1.

Let $f_{\Gamma_1}(t)$ denote the above first passage density. Rice proposed a two step numerical procedure for finding $f_{\Gamma_1}(t)$. First he derived a relation between $f_{\Gamma_1}(t)$ and the first recurrence time density function $r_{\Gamma_1}(t)$ which he defined as follows: given the occurrence of a negative crossing, crossing with negative slope, at time zero, $r_{\Gamma_1}(t)$ is the probability density for the random variable for the first positive crossing, crossing with positive slope, of b after zero. His relation is

$$f_{\Gamma}(t) = N^{-} \int_{t}^{\infty} r_{\Gamma}(\tau)\, d\tau \tag{2D}$$

where

$$N^{-} = -\int_{-\infty}^{0} \dot{y} \;\; p(b,\dot{y})\, d\dot{y}$$

He expressed $r_{\Gamma_1}(t)$ in a S. O. Rice exclusion series. Thus

$$r_{\Gamma_1}(t) = p_{+/-} \;\; (t/0) - \frac{1}{1!} \int_{0}^{t} p_{+,+/-}(s,t/0)\, ds + \tag{3D}$$

$$+ \frac{1}{2!} \int_{0}^{t} \int_{0}^{t} p_{+,+,+/-}(r,s,t/0)\, drds + \ldots$$

where $p_{+/-}(t/0)$, $p_{+,+/-}(s,t/0)$, $p_{+++/-}(r,s,t/0)$, etc. are defined as follows:

(a) $p_{+/-}(t/0)\, dt$ is the probability of having a positive crossing in the interval $(t, t + dt]$, given a negative crossing at zero.

(b) $p_{+,+/-}(s,t/0)\ dsdt$ is the probability of having positive crossings in the intervals $(s, s + ds]$ and $(t, t + dt]$, given a negative crossing at zero.

(c) $p_{+,+,+/-}(r,s,t/0)\ drdsdt$ is the probability of having positive crossings in the intervals $(r, r + dr]$, $(s, s + ds]$, and $(t, t + dt]$, given a negative crossing at zero.

$$\vdots$$

Rice made the assumption that the probability of occurrence of a positive crossing in an interval $(t, t + dt]$ depends only on the closest given prior crossing. From conditional probability, the following result may be obtained:

$$p_{+,+/-}(s,t/0) = p_{+/-,+}(t/0,s)\ p_{+/-}(s/0)$$

$$p_{+,+,+/-}(r,s,t/0) = p_{+/-,+,+}(t/0,r,s)\ p_{+/-,+}(s/0,r)\ p_{+/-}(s/0)$$

$$\vdots$$

Under Rice's assumption these functions simplify to

$$p_{+,+/-}(s,t,0) = p_{+/+}(t/s)\ p_{+/-}(s/0)$$

$$p_{+,+,+/-}(r,s,t/0) = p_{+/+}(t/s)\ p_{+/+}(s/r)\ p_{+/-}(r/0)$$

Rice used these simplifications to reduce the series to an integral equation. It is

$$p_{+1/-}(t/0) = r_{\Gamma_1}(t) + \int_0^t p_{+/+}(t/\eta)\, r_{\Gamma_1}(\eta)d\eta \qquad (4D)$$

He solved (4D) numerically for $r_{\Gamma_1}(t)$ and then used (2D) to obtain $f_{\Gamma_1}(t)$.

The functions $p_{+/-}(t/0)$ and $p_{+/+}(t/\eta)$ may be expressed in terms of $p(x,\dot{x})$ and $N^{+}(t/b,y,\eta)$. Thus

$$p_{+/-}(t/0) = \frac{1}{N^{-}} \int_{-\infty}^{0} N^{+}(t/b,\dot{y},0)\ |\dot{y}|\ p\ (b,\dot{y})d\dot{y}$$

$$p_{+/+}(t/\eta) = \frac{1}{N^{+}} \int_{0}^{\infty} N^{+}(t/b,\dot{y},\eta)\ \dot{y}\ p(b,\dot{y})\ d\dot{y}$$

Starting from equation (1D), substituting $g^{B}_{\Gamma_1}(\dot{y}/\eta)$ for $g_{\Gamma_1}(\dot{y}/\eta)$, letting $\dot{x}$ be negative, taking the limit as x tends to b from below, multiplying both sides by $\frac{|\dot{x}|\ p(b,\dot{x})}{N^{-}}$, and integrating on $\dot{x}$ from minus infinity to zero results in the following equation:

$$\frac{1}{N^-} \int_{-\infty}^{0} N^+(t/b,\dot{x},0)\ |\dot{x}|\ p(b,\dot{x})\ d\dot{x} =$$

$$\frac{1}{N^-}\int_{-\infty}^{0} \lim_{x\to b^-} f_{\Gamma_1}(x,\dot{x},t)\ \ |\dot{x}|\ p(b,\dot{x})\ d\dot{x}\ + \tag{5D}$$

$$+ \int_0^t \left[\int_0^\infty N^+(t/b,\dot{y},\eta)\ \dot{y}\ p(b,\dot{y})\ d\dot{y}\right] \frac{1}{N^-} \int_{-\infty}^{0} \lim_{x\to b^-} f_{\Gamma_1}(x,\dot{x},\eta)|\dot{x}|\ p(b,\dot{x})d\dot{x}d\eta$$

From conditional probability the integral

$$\frac{1}{N^-}\int_{-\infty}^{0} \lim_{x\to b^-} f_{\Gamma_1}(x,\dot{x},t)\ |\dot{x}|\ p(b,\dot{x})d\dot{x}$$

is $r_{\Gamma_1}(t)$. Thus, considering the expressions for $p_{+/+}(t/\eta)$ and $p_{+/-}(t/0)$, equation (5D) may be written

$$p_{+/-}(t/0) = r_{\Gamma_1}(t) + \int_0^t p_{+/+}(t/\eta)\ \ r_{\Gamma_1}(\eta)d\eta$$

which is J. R. Rices result.

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